Computers & Fluids 40 (2011) 149–155

Contents lists available at ScienceDirect

Computers & Fluids

journal homepage: www.elsevier.com/locate/compfluid

Three-dimensional discrete-velocity BGK model for the incompressible Navier–Stokes equations

Akinori Tamura *, Keita Okuyama, Shiro Takahashi, Masaya Ohtsuka

Energy and Environmental Systems Laboratory, Hitachi Ltd., 2-1, Omika-cho 7-chome, Hitachi-shi, Ibaraki-ken 319-1221, Japan

ARTICLE INFO

Article history:
Received 6 October 2009
Received in revised form 1 July 2010
Accepted 22 August 2010
Available online 27 August 2010

Keywords:
Lattice Boltzmann method
Finite difference Boltzmann method
Computational fluid dynamics
Three-dimensional model

ABSTRACT

The lattice Boltzmann method (LBM) has been widely used for the simulations of the incompressible Navier–Stokes (NS) equations. The finite difference Boltzmann method (FDBM) in which the discrete-velocity Boltzmann equation is solved instead of the lattice Boltzmann equation has also been applied as an alternative method for simulating the incompressible flows. The particle velocities of the FDBM can be selected independently from the lattice configuration. In this paper, taking account of this advantage, we present the discrete velocity Boltzmann equation that has a minimum set of the particle velocities with the lattice Bharnagar–Gross–Krook (BGK) model for the three-dimensional incompressible NS equations. To recover incompressible NS equations, tensors of the particle velocities have to be isotropic up to the fifth rank. Thus, we propose to apply the icosahedral vectors that have 13 degrees of freedom to the particle velocity distributions. Validity of the proposed model (D3Q13BGK) is confirmed by numerical simulations of the shear-wave decay problem and the Taylor–Green vortex problem. With respect to numerical accuracy, computational efficiency and numerical stability, we compare the proposed model with the conventional lattice BGK models (D3Q15, D3Q19 and D3Q27) and the multiple-relaxation-time (MRT) model (D3Q13MRT) that has the same degrees of freedom as our proposal. The comparisons show that the compressibility error of the proposed model is approximately double that of the conventional lattice BGK models, but the computational efficiency of the proposed model is superior to that of the others. The linear stability of the proposed model is also superior to that of the lattice BGK models. However, in non-linear simulations, the proposed model tends to be less stable than the others.

1. Introduction

The lattice Boltzmann method (LBM) [1–11] has been widely used for simulations of the incompressible Navier–Stokes (NS) equations. The simplest lattice Boltzmann equation is the lattice Bharnagar–Gross–Krook (BGK) equation which is based on a single-relaxation-time (SRT) approximation [12]. The multiple-relaxation-time (MRT) lattice Boltzmann equation that possesses many more degrees of freedom than the lattice BGK model was also developed as a numerically stable lattice Boltzmann model [10,11]. Due to use of different relaxation times, the MRT lattice Boltzmann equation overcomes some deficiencies of the lattice BGK equation, such as a fixed Prandtl number and a fixed ratio between the kinetic viscosity and the bulk viscosity. However computational efficiency of the MRT model is known to be inferior to that of the lattice BGK model due to computation of multiple relaxation processes. In spite of the above mentioned deficiencies, the lattice BGK model is the most popular lattice Boltzmann model due to its simplicity and good computational efficiency. In the LBM, the velocity distribution functions propagate from one node to adjacent nodes over the course of one time step. Therefore, the particle velocities of the LBM are limited to those that exactly link the lattice nodes.

The finite difference Boltzmann method (FDBM) [13–16] in which the discrete-velocity Boltzmann equation is solved instead of the lattice Boltzmann equation has also been applied as an alternative method for simulating the incompressible flows. The particle velocities of the FDBM can be selected independently from the lattice configuration due to use of the discrete-velocity Boltzmann equation [15,16]. In spite of this advantage, the particle velocity model developed for the LBM which needs many particle velocities is often used in simulations of the FDBM [13–15]. For computational efficiency, it is desirable to find the minimal set of particle velocities. In this paper, we present the discrete-velocity Boltzmann equation that has a minimum set of the particle velocities with the lattice BGK (SRT) model for the three-dimensional incompressible NS equations. It should be noted that the minimum set of the particle velocities with the MRT model has already been proposed by d'Humieres et al. [10]. This model has 13 particle velocities (D3Q13MRT) whose degrees of freedom are the same as those of the particle velocities of our proposal (D3Q13BGK). We compare

* Corresponding author. Tel.: +81 294 55 9429; fax: +81 294 55 9973.
E-mail address: akinori.tamura.mt@hitachi.com (A. Tamura).

0045-7930/$ - see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.compfluid.2010.08.019

the proposed model, with respect to numerical accuracy, computational efficiency and numerical stability, with the commonly used lattice BGK models (D3Q15, D3Q19 [7] and D3Q27 [8]) and the D3Q13MRT model [10].

2. Discrete-velocity BGK model

In this section, we present the discrete-velocity Boltzmann equation that has a minimum set of particle velocities with the lattice BGK model for the three-dimensional incompressible NS equations:

$$\frac{\partial u_\beta}{\partial x_\beta} = 0, \tag{1}$$

$$\frac{\partial u_\alpha}{\partial t} + u_\beta \frac{\partial u_\alpha}{\partial x_\beta} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_\alpha} + \nu \frac{\partial^2 u_\alpha}{\partial x_\beta^2}, \tag{2}$$

where t and x_α are the time and the spatial coordinate. The subscripts α and β are the number of the spatial coordinates and the summation convention is applied to these subscripts. u_α, p, ρ_0 and ν are the flow velocity in the x_α direction, the pressure, the reference density and the kinetic viscosity, respectively.

We let $c_{i\alpha}$ be the particle velocity of the ith particle in the x_α direction, where $i = 1,2,\ldots,I$ and I represents the number of particle velocities. The subscript i represents the kind of particles and the summation convention is not applied to this subscript. $f_i(x_\alpha,t)$ is the velocity distribution function. The macroscopic variables ρ and u_α are defined as

$$\rho = \sum_{i=1}^{I} f_i, \tag{3}$$

$$u_\alpha = \frac{1}{\rho}\sum_{i=1}^{I} f_i c_{i\alpha}. \tag{4}$$

The equation to be solved in the FDBM is the following discrete-velocity BGK equation [14–16]:

$$\frac{\partial f_i}{\partial t} + c_{i\beta}\frac{\partial f_i}{\partial x_\beta} = \frac{f_i^{eq}(\rho, u_\alpha) - f_i}{\tau}, \tag{5}$$

where the relaxation time τ is inversely proportional to density. τ is a constant for constant density flows. The local equilibrium velocity distribution function $f_i^{eq}(\rho, u_\alpha)$ is a given function of the macroscopic variables. The MRT model can also be applied to the collision process of Eq. (5) instead of the lattice BGK (SRT) model.

A non-dimensional expression is convenient for the following discussions and numerical simulations. We let c_0 and L be the reference speed and length, respectively. The non-dimensional variables are defined as follows:

$$\hat{t} = \frac{c_0}{L}t, \quad \hat{x}_\alpha = \frac{x_\alpha}{L}, \quad \hat{c}_{i\alpha} = \frac{c_{i\alpha}}{c_0}, \quad \hat{f}_i = \frac{f_i}{\rho_0}, \quad \hat{f}_i^{eq} = \frac{f_i^{eq}}{\rho_0},$$
$$\hat{\rho} = \frac{\rho}{\rho_0}, \quad \hat{u}_\alpha = \frac{u_\alpha}{c_0}, \quad \hat{p} = \frac{p}{\rho_0 c_0^2}, \quad \text{and} \quad \hat{\nu} = \frac{\nu}{c_0 L}. \tag{6}$$

From Eqs. (3) and (4), the non-dimensional density and flow velocity are defined as

$$\hat{\rho} = \sum_{i=1}^{I} \hat{f}_i, \tag{7}$$

$$\hat{u}_\alpha = \frac{1}{\hat{\rho}}\sum_{i=1}^{I} \hat{f}_i \hat{c}_{i\alpha}. \tag{8}$$

The non-dimensional discrete-velocity BGK equation is defined as

$$\frac{\partial \hat{f}_i}{\partial \hat{t}} + \hat{c}_{i\beta}\frac{\partial \hat{f}_i}{\partial \hat{x}_\beta} = \frac{\hat{f}_i^{eq}(\hat{\rho}, \hat{u}_\alpha) - \hat{f}_i}{\varepsilon}, \tag{9}$$

where $\varepsilon = c_0\tau/L$ is the non-dimensional parameter proportional to the Knudsen number.

In order to clarify the constraints of the local equilibrium distribution function for the incompressible NS equations, we employ the Chapman–Enskog expansion [17] for $\varepsilon \ll 1$, which is essentially a formal multi-scaling expansion. The time derivative is expanded as

$$\frac{\partial}{\partial \hat{t}} = \frac{\partial}{\partial \hat{t}_0} + \varepsilon\frac{\partial}{\partial \hat{t}_1} + \cdots. \tag{10}$$

The velocity distribution function is also expanded about $\hat{f}_i^{eq}$,

$$\hat{f}_i = \hat{f}_i^{eq} + \varepsilon \hat{f}_i^{(1)} + \varepsilon^2 \hat{f}_i^{(2)} \cdots. \tag{11}$$

$\hat{f}_i^{eq}$ depends on the local macroscopic variables and should satisfy the following constraints:

$$\sum_{i=1}^{I} \hat{f}_i^{eq} = \hat{\rho}, \tag{12}$$

$$\sum_{i=1}^{I} \hat{f}_i^{eq}\hat{c}_{i\alpha} = \hat{\rho}\hat{u}_\alpha. \tag{13}$$

$\varepsilon \hat{f}_i^{(1)} + \varepsilon^2 \hat{f}_i^{(2)} + O(\varepsilon^3)$ is the non equilibrium velocity distribution function which has the following constraints:

$$\sum_{i=1}^{I} \hat{f}_i^{(l)} = 0, \tag{14}$$

$$\sum_{i=1}^{I} \hat{f}_i^{(l)}\hat{c}_{i\alpha} = 0, \tag{15}$$

where $l = 1,2,\ldots$. Substituting the above expansions into the discrete-velocity BGK Eq. (9), we find

$$\frac{\partial \hat{f}_i^{eq}}{\partial \hat{t}_0} + \hat{c}_{i\beta}\frac{\partial \hat{f}_i^{eq}}{\partial \hat{x}_\beta} + \varepsilon\left(\frac{\partial \hat{f}_i^{(1)}}{\partial \hat{t}_0} + \frac{\partial \hat{f}_i^{eq}}{\partial \hat{t}_1} + \hat{c}_{i\beta}\frac{\partial \hat{f}_i^{(1)}}{\partial \hat{x}_\beta}\right) = -\hat{f}_i^{(1)} - \varepsilon \hat{f}_i^{(2)}, \tag{16}$$

up to $O(\varepsilon)$.

Multiplying Eq. (16) by unity and summing up all the particles, we obtain the continuity or mass conservation equation:

$$\frac{\partial \hat{\rho}}{\partial \hat{t}_0} + \varepsilon\frac{\partial \hat{\rho}}{\partial \hat{t}_1} + \frac{\partial \hat{\rho}\hat{u}_\beta}{\partial \hat{x}_\beta} = 0. \tag{17}$$

Multiplying Eq. (16) by c_α and summing up all the particles, we obtain the following equation:

$$\frac{\partial \hat{\rho}\hat{u}_\alpha}{\partial \hat{t}_0} + \varepsilon\frac{\partial \hat{\rho}\hat{u}_\alpha}{\partial \hat{t}_1} + \frac{\partial}{\partial \hat{x}_\beta}\sum_{i=1}^{I} \hat{f}_i^{eq}\hat{c}_{i\alpha}\hat{c}_{i\beta} + \varepsilon\frac{\partial}{\partial \hat{x}_\beta}\sum_{i=1}^{I} \hat{f}_i^{(1)}\hat{c}_{i\alpha}\hat{c}_{i\beta} = 0. \tag{18}$$

To recover the momentum equation from Eq. (18), the following constraint is imposed on $\hat{f}_i^{eq}$:

$$\sum_{i=1}^{I} \hat{f}_i^{eq}\hat{c}_{i\alpha}\hat{c}_{i\beta} = \hat{\rho}\hat{c}_s^2\delta_{\alpha\beta} + \hat{\rho}\hat{u}_\alpha\hat{u}_\beta, \tag{19}$$

where $\hat{c}_s = c_s/c_0$ is non-dimensional speed of sound. The last term of Eq. (18) has to correspond to a viscous term. Using Eqs. (16) and (19), we can rewrite the last term of Eq. (18) as

$$\varepsilon\frac{\partial}{\partial \hat{x}_\beta}\sum_{i=1}^{I} \hat{f}_i^{(1)}\hat{c}_{i\alpha}\hat{c}_{i\beta} = -\varepsilon\frac{\partial}{\partial \hat{x}_\beta}\left(\frac{\partial}{\partial \hat{t}_0}\left(\hat{\rho}\hat{c}_s^2\delta_{\alpha\beta} + \hat{\rho}\hat{u}_\alpha\hat{u}_\beta\right) + \frac{\partial}{\partial \hat{x}_\gamma}\sum_{i=1}^{I} \hat{f}_i^{eq}\hat{c}_{i\alpha}\hat{c}_{i\beta}\hat{c}_{i\gamma}\right) + O(\varepsilon^2). \tag{20}$$

The time derivatives are transferred to space derivatives using the Euler equations which are obtained from Eq. (16). With some algebra, we can finally obtain the following equation: