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Technical note

Absorbing boundary condition for nonlinear Euler equations in primitive variables based on the Perfectly Matched Layer technique

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ABSTRACT

For aeroacoustics problems, the nonlinear Euler equations are often written in primitive variables in which the pressure is treated as a solution variable. In this paper, absorbing boundary conditions based on the Perfectly Matched Layer (PML) technique are presented for nonlinear Euler equations in primitive variables. A pseudo mean flow is introduced in the derivation of the PML equations for increased efficiency. Absorbing equations are presented in unsplit physical primitive variables in both the Cartesian and cylindrical coordinates. Numerical examples show the effectiveness of the proposed equations although they are not theoretically perfectly matched to the nonlinear Euler equations. The derived equations are tested in numerical examples and compared with the PML absorbing boundary condition in conservation form that was formulated in an earlier work. The performance of the PML in primitive variables is found to be close to that of the conservation formulation. A comparison with the linear PML in nonlinear problems is also considered. It is found that using nonlinear absorbing equations presented in this paper significantly improves the performance of the absorbing boundary condition for strong nonlinear cases.

1. Introduction

Perfectly Matched Layer (PML) is a technique of developing non-reflecting boundary conditions. Similar to the buffer zone and sponge layer techniques [1,2], extra absorbing zones are added, in which numerical solutions are damped [3]. However, PML zones are usually much thinner compared to most other buffer zones, as the absorbing zone is theoretically reflectionless for multi-dimensional linear waves of any angle and frequency. Berenger proposed firstly the PML for Maxwell's equations to absorb the electromagnetic wave at open boundaries in 1994 [4]. Berenger's technique was first applied to the linear Euler equations with a uniform mean flow for the field of acoustics in 1996 [5]. Since then, many efforts have been made in the study of the PML technique and to extend its application and improve its performance. Some recent advances in the development of PML as absorbing boundary conditions were reviewed in Ref. [3]. In recent years, many progresses have been made in the development of PML for Computational Fluid Dynamics (CFD) and Computational Aeroacoustics (CAA). It started with the cases for the linearized Euler equations from constant mean flows to non-uniform mean flows [5–11], then extended to the cases for the fully nonlinear Euler equations [12]. And the applications of PML to linearized Navier–Stokes equations [13] and nonlinear Navier–Stokes equations [14] have been discussed. Recently, PML for the fully nonlinear Euler and Navier–Stokes equations has been given in Ref. [15]. And in addition, PML equations were developed to accommodate the uniform mean flow in an arbitrary direction [16].

In our previous study of PML for nonlinear problems, the Euler and Navier–Stokes equations were given in the conservation form [15]. In the present paper, a PML absorbing boundary condition for the nonlinear Euler equations in primitive variables is developed. This is motivated by the observation that the nonlinear Euler equations in primitive variables remain a popular form of the governing equations for inviscid compressible flows, especially for many nonlinear aeroacoustics problems. The purpose of this paper is two-fold. First, new absorbing boundary conditions in primitive variables based on the PML technique are proposed for both the Cartesian and cylindrical coordinate systems. Second, comparisons in the performance with the nonlinear PML in the conservation form and the linear PML in primitive variables are conducted, to demonstrate the accuracy and necessity of proposed new set of boundary conditions. To deal with the nonlinear terms involving spatial derivatives and to facilitate the application of PML complex change of variables in frequency domain, new auxiliary variables are introduced. The final form of the absorbing equations is presented in unsplit physical primitive variables. Numerical examples

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are presented to demonstrate the validity and efficiency of the proposed boundary conditions. In the next section, PML equations for the nonlinear Euler equations in primitive variables are derived. It follows a three-step method proposed in Refs. [10,15]. Firstly, a proper space–time transformation is applied to the governing equations, so that linear waves have consistent phase and group velocities. Secondly, a PML complex change of variables is applied in the frequency domain. And thirdly, the time domain PML absorbing boundary condition is derived by a conversion of the frequency domain equations to the time domain equations. Numerical examples that validate the efficiency and validity of the proposed PML equations are presented in Section 3. Concluding remarks are given in Section 4.

2. PML equations in primitive variables

2.1. Cartesian coordinates

The two-dimensional nonlinear Euler equations in primitive variables are written in the Cartesian coordinate system as

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{u}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{u}}{\partial y} = 0 \tag{1}$$

where

$$\mathbf{u} = \begin{bmatrix} \rho \\ u \\ v \\ p \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} u & \rho & 0 & 0 \\ 0 & u & 0 & 1/\rho \\ 0 & 0 & u & 0 \\ 0 & \gamma p & 0 & u \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} v & 0 & \rho & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & v & 1/\rho \\ 0 & 0 & \gamma p & v \end{bmatrix}$$

where u and v are the velocity components in x and y directions, respectively, p is the pressure, ρ is the density. In this paper, the velocity is nondimensionalized by a reference speed of sound c_∞, density by a reference density ρ_∞ and pressure by $\rho_\infty c_\infty^2$.

We wish to formulate absorbing equations so that out-going disturbances can be exponentially reduced once they enter a PML domain while causing as little numerical reflection as possible. For nonlinear Euler equations, the solutions can often be partitioned into two parts. One part is the time-independent mean state, the other part is the time-dependent fluctuation. It would be efficient to absorb only the time-dependent fluctuations in the PML domains. When the mean flow is unknown, an approximate mean flow or a pseudo mean flow can be used in the formulation as in Ref. [17]. Therefore, we express the primitive variables inside a PML domain as

$$\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}' \tag{2}$$

where the superscript "bar" indicates the mean flow (or the pseudo mean flow), and the superscript "prime" indicates the difference between the mean flow and the actual flow. The mean flow or pseudo mean flow should satisfy the steady Euler equations,

$$\overline{\mathbf{A}}\frac{\partial \bar{\mathbf{u}}}{\partial x} + \overline{\mathbf{B}}\frac{\partial \bar{\mathbf{u}}}{\partial y} = 0 \tag{3}$$

According to Eqs. (1)–(3), we can get the following equation for the time dependent part of the solution:

$$\frac{\partial \mathbf{u}'}{\partial t} + \mathbf{A}\frac{\partial \mathbf{u}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{u}}{\partial y} - \overline{\mathbf{A}}\frac{\partial \bar{\mathbf{u}}}{\partial x} - \overline{\mathbf{B}}\frac{\partial \bar{\mathbf{u}}}{\partial y} = 0 \tag{4}$$

We shall derive absorbing equations for Eq. (4). To facilitate the derivation, Eq. (4) can be rewritten as

$$\begin{aligned} \frac{\partial \mathbf{u}'}{\partial t} &+ \overline{\mathbf{A}}\frac{\partial (\mathbf{u}-\bar{\mathbf{u}})}{\partial x} + (\mathbf{A}-\overline{\mathbf{A}})\frac{\partial \bar{\mathbf{u}}}{\partial x} + (\mathbf{A}-\overline{\mathbf{A}})\frac{\partial (\mathbf{u}-\bar{\mathbf{u}})}{\partial x} + \overline{\mathbf{B}}\frac{\partial (\mathbf{u}-\bar{\mathbf{u}})}{\partial y} \\ &+ (\mathbf{B}-\overline{\mathbf{B}})\frac{\partial \bar{\mathbf{u}}}{\partial y} + (\mathbf{B}-\overline{\mathbf{B}})\frac{\partial (\mathbf{u}-\bar{\mathbf{u}})}{\partial y} = 0 \end{aligned} \tag{5}$$

For the stability of the PML, a space–time transformation $\bar{t} = t + \beta x$ is necessary in the derivation process, where β is a parameter dependent on the mean flow profile, as discussed in Refs. [10,15]. Here, a mean flow that is dominantly in the x direction is assumed. In transformed coordinates, Eq. (5) can be written as follows:

$$\begin{aligned} \frac{\partial \mathbf{u}'}{\partial \bar{t}} &+ \beta\overline{\mathbf{A}}\frac{\partial (\mathbf{u}-\bar{\mathbf{u}})}{\partial \bar{t}} + \beta(\mathbf{A}-\overline{\mathbf{A}})\frac{\partial (\mathbf{u}-\bar{\mathbf{u}})}{\partial \bar{t}} + \overline{\mathbf{A}}\frac{\partial (\mathbf{u}-\bar{\mathbf{u}})}{\partial x} \\ &+ (\mathbf{A}-\overline{\mathbf{A}})\frac{\partial (\mathbf{u}-\bar{\mathbf{u}})}{\partial x} + (\mathbf{A}-\overline{\mathbf{A}})\frac{\partial \bar{\mathbf{u}}}{\partial x} + \overline{\mathbf{B}}\frac{\partial (\mathbf{u}-\bar{\mathbf{u}})}{\partial y} + (\mathbf{B}-\overline{\mathbf{B}})\frac{\partial (\mathbf{u}-\bar{\mathbf{u}})}{\partial y} \\ &+ (\mathbf{B}-\overline{\mathbf{B}})\frac{\partial \bar{\mathbf{u}}}{\partial y} = 0 \end{aligned} \tag{6}$$

In the frequency domain, we have

$$\begin{aligned} &(-i\omega)\tilde{\mathbf{u}}' + (-i\omega)\beta\overline{\mathbf{A}}(\widetilde{\mathbf{u}-\bar{\mathbf{u}}}) + \beta(\widetilde{\mathbf{A}-\overline{\mathbf{A}}}) * (-i\omega)(\widetilde{\mathbf{u}-\bar{\mathbf{u}}}) \\ &\quad + \overline{\mathbf{A}}\frac{\partial (\widetilde{\mathbf{u}-\bar{\mathbf{u}}})}{\partial x} + (\widetilde{\mathbf{A}-\overline{\mathbf{A}}}) * \frac{\partial (\widetilde{\mathbf{u}-\bar{\mathbf{u}}})}{\partial x} + (\widetilde{\mathbf{A}-\overline{\mathbf{A}}})\frac{\partial \bar{\mathbf{u}}}{\partial x} \\ &\quad + \overline{\mathbf{B}}\frac{\partial (\widetilde{\mathbf{u}-\bar{\mathbf{u}}})}{\partial y} + (\widetilde{\mathbf{B}-\overline{\mathbf{B}}}) * \frac{\partial (\widetilde{\mathbf{u}-\bar{\mathbf{u}}})}{\partial y} + (\widetilde{\mathbf{B}-\overline{\mathbf{B}}})\frac{\partial \bar{\mathbf{u}}}{\partial y} = 0 \end{aligned} \tag{7}$$

where a tilde indicates the time Fourier transformed variable and $*$ denotes convolution integral.

By applying the PML complex change of variables to Eq. (7), we get

$$\begin{aligned} &(-i\omega)\tilde{\mathbf{u}}' + (-i\omega)\beta\overline{\mathbf{A}}(\widetilde{\mathbf{u}-\bar{\mathbf{u}}}) + \beta(\widetilde{\mathbf{A}-\overline{\mathbf{A}}}) * (-i\omega)(\widetilde{\mathbf{u}-\bar{\mathbf{u}}}) \\ &\quad + \overline{\mathbf{A}}\frac{\partial (\widetilde{\mathbf{u}-\bar{\mathbf{u}}})}{\partial x}\frac{1}{1+\frac{i\sigma_x}{\omega}} + (\widetilde{\mathbf{A}-\overline{\mathbf{A}}}) * \frac{\partial (\widetilde{\mathbf{u}-\bar{\mathbf{u}}})}{\partial x}\frac{1}{1+\frac{i\sigma_x}{\omega}} \\ &\quad + (\widetilde{\mathbf{A}-\overline{\mathbf{A}}})\frac{\partial \bar{\mathbf{u}}}{\partial x} + \overline{\mathbf{B}}\frac{\partial (\widetilde{\mathbf{u}-\bar{\mathbf{u}}})}{\partial y}\frac{1}{1+\frac{i\sigma_y}{\omega}} \\ &\quad + (\widetilde{\mathbf{B}-\overline{\mathbf{B}}}) * \frac{\partial (\widetilde{\mathbf{u}-\bar{\mathbf{u}}})}{\partial y}\frac{1}{1+\frac{i\sigma_y}{\omega}} + (\widetilde{\mathbf{B}-\overline{\mathbf{B}}})\frac{\partial \bar{\mathbf{u}}}{\partial y} = 0 \end{aligned} \tag{8}$$

where σ_x and σ_y are absorption coefficients, which are positive and could be functions of x and y, respectively [5].

To rewrite the above equation in the time domain, we introduce auxiliary variables $\mathbf{q}_1$ and $\mathbf{q}_2$ as

$$(-i\omega)\tilde{\mathbf{q}}_1 = \frac{\partial (\widetilde{\mathbf{u}-\bar{\mathbf{u}}})}{\partial x}\frac{1}{1+\frac{i\sigma_x}{\omega}} + (-i\omega)\beta(\widetilde{\mathbf{u}-\bar{\mathbf{u}}}) \tag{9}$$

$$(-i\omega)\tilde{\mathbf{q}}_2 = \frac{\partial (\widetilde{\mathbf{u}-\bar{\mathbf{u}}})}{\partial y}\frac{1}{1+\frac{i\sigma_y}{\omega}} \tag{10}$$

Then Eq. (8) becomes

$$\begin{aligned} &(-i\omega)\tilde{\mathbf{u}}' + \overline{\mathbf{A}}(-i\omega)\tilde{\mathbf{q}}_1 + (\widetilde{\mathbf{A}-\overline{\mathbf{A}}}) * (-i\omega)\tilde{\mathbf{q}}_1 + \overline{\mathbf{B}}(-i\omega)\tilde{\mathbf{q}}_2 \\ &\quad + (\widetilde{\mathbf{B}-\overline{\mathbf{B}}}) * (-i\omega)\tilde{\mathbf{q}}_2 + (\widetilde{\mathbf{A}-\overline{\mathbf{A}}})\frac{\partial \bar{\mathbf{u}}}{\partial x} + (\widetilde{\mathbf{B}-\overline{\mathbf{B}}})\frac{\partial \bar{\mathbf{u}}}{\partial y} = 0 \end{aligned} \tag{11}$$

It is easy to get the time domain equations for $\mathbf{q}_1$ and $\mathbf{q}_2$ in the original space–time domain as

$$\frac{\partial \mathbf{q}_1}{\partial t} + \sigma_x\mathbf{q}_1 = \frac{\partial (\mathbf{u}-\bar{\mathbf{u}})}{\partial x} + \sigma_x\beta(\mathbf{u}-\bar{\mathbf{u}}) \tag{12}$$

$$\frac{\partial \mathbf{q}_2}{\partial t} + \sigma_y\mathbf{q}_2 = \frac{\partial (\mathbf{u}-\bar{\mathbf{u}})}{\partial y} \tag{13}$$

Rewriting Eq. (11) back into the original space–time domain, we get

$$\begin{aligned} \frac{\partial \mathbf{u}'}{\partial t} &+ \overline{\mathbf{A}}\frac{\partial \mathbf{q}_1}{\partial t} + (\mathbf{A}-\overline{\mathbf{A}})\frac{\partial \mathbf{q}_1}{\partial t} + \overline{\mathbf{B}}\frac{\partial \mathbf{q}_2}{\partial t} + (\mathbf{B}-\overline{\mathbf{B}})\frac{\partial \mathbf{q}_2}{\partial t} \\ &+ (\mathbf{A}-\overline{\mathbf{A}})\frac{\partial \bar{\mathbf{u}}}{\partial x} + (\mathbf{B}-\overline{\mathbf{B}})\frac{\partial \bar{\mathbf{u}}}{\partial y} = 0 \end{aligned} \tag{14}$$