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Well-balanced RKDG2 solutions to the shallow water equations over irregular domains with wetting and drying

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ABSTRACT

This paper presents a new one-dimensional (1D) second-order Runge–Kutta discontinuous Galerkin (RKDG2) scheme for shallow flow simulations involving wetting and drying over complex domain topography. The shallow water equations that adopt water level (instead of water depth) as a flow variable are solved by an RKDG2 scheme to give piecewise linear approximate solutions, which are locally defined by an average coefficient and a slope coefficient. A wetting and drying technique proposed originally for a finite volume MUSCL scheme is revised and implemented in the RKDG2 solver. Extra numerical enhancements are proposed to amend the local coefficients associated with water level and bed elevation in order to maintain the well-balanced property of the RKDG2 scheme for applications with wetting and drying. Friction source terms are included and evaluated using splitting implicit discretization, implemented with a physical stopping condition to ensure stability. Several steady and unsteady benchmark tests with/without friction effects are considered to demonstrate the performance of the present model.

1. Introduction

Free-surface flows with a vertical scale that is much smaller than the horizontal dimensions may be mathematically described by the shallow water equations (SWE). In recent years, there have been increasing interests in developing robust numerical schemes for solving the shallow water equations for different engineering purposes [1–10,15–17,19–28,30–36,40–53] and Godunov-type schemes [18] have experienced a vigorous development in the last two decades (see [30,33,41] for informative reviews). A Godunov-type method generally solves the conservative form of the SWE and introduces a hyperbolic wave pattern to the discretization scheme (*i.e.* approximate Riemann solver [31]) to compute inter-cell fluxes. Numerical oscillations appearing in those high-order schemes [18,31] are controlled by incorporating a slope limiting procedure [37]. To cope with complex domain topography in practical simulations, various mathematical and numerical techniques have been proposed to effectively discretize the bed gradient source terms and achieve well-balanced schemes (*e.g.* [1,2,22,24,34,26,27,42] and the references therein). Many real-world applications also require a model to be able to handle repeated wetting and drying over irregular domain topography [3–10,15–17,19,21,23,25,35,36] and to correctly represent friction effects [9,35,40,52]. These are the current active research topics in computational hydraulics and are investigated in this work in the context of a discontinuous Galerkin Godunov-type scheme.

It is not trivial to design a numerical approach to handling wetting and drying as it is essentially a moving boundary problem where the wet/dry interface continuously evolves along the problem domain. Various techniques have been reported to model wetting and drying, mainly in the family of the finite volume methods. Brufau et al. [6,7] presented a numerical technique that temporarily modifies ground elevation to approximate wetting and drying for both steady [6] and unsteady flows [7]. In [7], a numerical technique was implemented to control negative depth and eliminate the mass error by locally modifying the flow variables in those cells with negative water depth (the flow variables in the direct neighbours of these cells might be also affected). Audusse et al. [1] initiated the work towards a general strategy for wetting and drying that avoids effective but sophisticated numerical treatments. The authors' method of hydrostatic reconstruction is becoming increasingly popular and been extended by many other researchers (*e.g.* [4,25,26]). Their work also stimulated another wetting and drying technique proposed by Liang and Marche [35]. Other finite volume wetting and drying algorithms include the work by Begnudelli and Sanders [3], Casulli [10] and Nicolos and Delis [23]. Efforts have also been made to include friction effects in a wetting and drying algorithm for more practical hydraulic simulations (*e.g.* [9,35,36,40,52]).

Runge–Kutta (RK) discontinuous Galerkin (DG) methods have recently gained popularity in solving the SWE [5,8,15,20–22,28,32,43–49] due to their many advantages over the spectral

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finite volume (*e.g.* WENO schemes [55,57]) and the traditional finite element methods (*i.e.* continuous Galerkin method [56]). A DG space discretization naturally combines the well-established finite element theory (*i.e.* locally employing compactly-supported shape functions) with those desirable finite volume tools (*e.g.* slope limiter, Godunov-type fluxes) for fluid computation [11,12,14,32]. It converts the system of conservation laws (*e.g.* the SWE) into a finite number of time-dependent systems of ordinary differential equations (ODE), which are then integrated by an explicit Runge–Kutta (RK) time integration method, in order to *locally* store and evolve the finite element degrees of freedom [11–14]. The solution may be of an arbitrary order of polynomial approximation in each element and fall discontinuous over inter-elemental boundaries. Therefore, RKDG methods preserve mass perfectly as the finite volume methods do, and meanwhile, have further appealing properties, *e.g.* ease to achieve high-order accuracy [12], scalability for parallel implementation [14], straightforward setup towards a well-balanced scheme [22,32] and suitability for adaptive discretizations with *hp*-refinement [46,47]. However, a main disadvantage, compared with the conventional finite element and spectral methods, is that the RKDG methods normally involve a larger number of degrees of freedom, which consequently demands higher computational costs and stricter stability requirement based on the CFL condition (see [54–57] for details). This however is compensated by the better convergence property and the inherent local structure of these methods, which facilities parallel computation.

Few attempts have been devoted to the issue of wetting and drying in the context of a DG local approximation. Bokhove [5] used a transient moving-mesh method to locate the wet/dry interface. Ern et al. [15] proposed a less complicated fixed-mesh approach based on a slope modification technique. But like the scheme presented by Brufau et al. [7], the method required addition of water mass (*i.e.* removal of negative water depth) to ensure depth positivity. Another slope adaptation technique, which conserves mass, was recently delivered by Bunya et al. [8] by employing the thin water layer approach (*i.e.* to introduce small water depth in the dry cells to prevent direct calculation of wet/dry interface). The technique was also adopted by Gourgue et al. [21]. However, introduction of thin water depth into the computation essentially violates the momentum conservation and may therefore degrade the accuracy of a numerical method [8,31] (this approach also requires a careful treatment to avoid unphysical fluxes in dry areas [8]). To the current authors' best knowledge, no attention has yet been paid to the discretization of friction source terms in the context of an RKDG method involving wetting and drying.

In this paper, an alternative wetting and drying algorithm, which further considers the friction effects, is designed for a second-order RKDG model (RKDG2) [20,22]. The non-negative reconstruction of the Riemann states, suggested by Liang and Marche [35] for a MUSCL-type scheme [50] is extended to the RKDG2 framework. In order to preserve the well-balanced property in the presence of wet/dry fronts, extra local amendments are made to the linear projection of the topographic data [22] and the coefficients defining the RDKG2 local linear approximate solution. The friction source terms are discretized using a splitting implicit approach that independently applies to both the average and the slope coefficients. The new RKDG2 scheme provides accurate predictions for frictional flows over complex topographies with moving wet/dry interfaces and ensures non-negative water depth and mass conservation.

2. Shallow water equations (SWE)

The SWE for long-wave propagation may be derived by integrating in depth the 3D Reynolds averaged Navier–Stokes equations by assuming negligible vertical particle acceleration and thus hydrostatic distribution. Including bed slope and friction effects the SWE may be adequate for describing a wide range of 1D and 2D shallow flow problems. In recent years, it has been generally accepted that the use of the surface water elevation $\eta(x,t)$ instead of the water depth $h(x,t)$ as a flow variable in the mathematical shallow water model may lead to a well-balanced numerical scheme that preserves the solution of lake at rest at the computational level [24,26,34,35]. Furthermore, adopting η as a flow variable improves the quality of a slope limiting process for RKDG methods [22,32]. In a matrix form, the 1D conservation laws of the nonlinear hyperbolic SWE may be written as [24,35]

$$\mathbf{U}_t + \mathbf{F}_x = \mathbf{S} \tag{1}$$

In which, $\mathbf{U} = [\eta, q]^T$. $q(x,t)$ is the unit-width discharge. $\eta(x,t) = h(x,t) + z(x)$ with $z(x)$ being the ground elevation. $u(x,t) = q(x,t)/h(x,t)$ is the depth-averaged velocity. t and x denotes, respectively, the time and space coordinates. $\mathbf{F} = [q, q^2/h + g(\eta^2 - 2\eta z)/2]^T$ is the flux vector such that $\mathbf{J} = \partial\mathbf{F}/\partial\mathbf{U}$ has two real eigenvalues $\lambda^{1,2} = u \pm c$ and two associated real eigenvectors $\mathbf{e}^{1,2} = [1, \lambda^{1,2}]^T$, where $c = \sqrt{gh}$ is the shallow wave speed and g is the gravitational acceleration. Obviously, the system is strictly hyperbolic if $h \neq 0$ [31]. $\mathbf{S} = \mathbf{S_b} + \mathbf{S_f}$ is the vector containing the bottom topography and friction source terms. $\mathbf{S_b} = [0, S_b]^T$ with $S_b = g\eta S_0$ and $S_0 = -\partial z/\partial x$. $\mathbf{S_f} = [0, S_f]^T$, where $S_f = -C_f u|u|$ with $C_f = gn_M^2/h^{1/3}$ and n_M being the Manning coefficient.

3. RKDG2 scheme with wetting and drying

In this section, a general review of the RKDG2 scheme is first presented for solving the 1D SWE. Then the new wetting and drying algorithm is proposed, followed by the discretization of the friction source terms.

3.1. An overview of the RKGD2 method

The 1D domain, on which the governing equations are solved, has a length of L and is divided by the interface points $0 = x_{1/2} < x_{3/2} < \cdots < x_{N+1/2} = L$ into N uniform intervals (cells). The size of an arbitrary cell $I_i = [x_{i-1/2}; x_{i+1/2}]$ is $\Delta x = x_{i+1/2} - x_{i-1/2}$ and the nodal point is at $x_i = (x_{i+1/2} + x_{i-1/2})/2$. When solving the conservation laws of the SWE (1) using a finite element Galerkin method, a kth order approximation of the flow variables $\mathbf{U_h}(x,t) = [\eta_h(x,t), q_h(x,t)]^T$ is sought, which belongs to the finite dimensional space $V_h = \{p : p|_{I_i} \in P^k(I_i)\}$ and $P^k(I_i)$ is the polynomial space in I_i of degree at most k [13]. The approximation gives $(k+1)$th order of accuracy in space. In order to derive the RKDG discretized governing equations, a test function $v_h \in V_h$ is introduced to (1), which is then integrated over I_i. Subsequently, integrating by part the flux derivative term gives a weak form to (1)

$$\int_{I_i} \partial_t \mathbf{U_h}(x,t)\, v_h(x)dx + \Bigg[\mathbf{F}(\mathbf{U_h}(x_{i+1/2},t))\, v_h(x_{i+1/2}) - \mathbf{F}(\mathbf{U_h}(x_{i-1/2},t))\, v_h(x_{i-1/2}) - \int_{I_i} \mathbf{F}(\mathbf{U_h}(x,t))\, v_h(x)dx\Bigg] = \int_{I_i} \mathbf{S}(\mathbf{U_h}(x,t))\, v_h(x)dx \tag{2}$$

A finite number of local basis functions are introduced using the L^2-orthogonal basis of Legendre polynomials [12] on I_i to locally expand the flow variables. The expanded flow variables are then substituted into the weak formulation (2) and a test function is chosen to specifically coincide with a basis function. As a result the DG space discretisation of (1) reduces to an independent system of ODEs for the expansion coefficients [22]. In this work,