Computers & Fluids 46 (2011) 231–236

Contents lists available at ScienceDirect

Computers & Fluids

journal homepage: www.elsevier.com/locate/compfluid

An implicit MacCormack scheme for unsteady flow calculations

J. Fürst *, P. Furmánek

Faculty of Mechanical Engineering, Czech Technical University in Prague, Karlovo nam. 13, 121 35 Praha, Czech Republic

A R T I C L E I N F O

Article history:
Received 30 April 2010
Received in revised form 21 September 2010
Accepted 27 September 2010
Available online 8 October 2010

Keywords:
FVM
Implicit method
Arbitrary Lagrangian–Eulerian method
Unsteady flows

A B S T R A C T

This paper describes the implicit MacCormack scheme [1] in finite volume formulation. Unsteady flows with moving boundaries are considered using arbitrary Lagrangian–Eulerian approach.

The scheme is unconditionally stable and does not require solution of large systems of linear equations. Moreover, the upgrade from explicit MacCormack scheme to implicit one is very simple and straightforward.

Several computational results for 2D and 3D flows over profiles and wings are presented for the case of inviscid and viscous flows.

1. Introduction

The explicit MacCormack scheme equipped with proper artificial viscosity terms proved very good accuracy and efficiency in many industrial applications, especially in the case of inviscid compressible flows. It has been applied successfully for calculations of transonic flows over profiles and wings [2], or through turbine cascades [3].

The main drawback of the explicit scheme is its time-step limitation due to stability condition. It becomes more important for unsteady flows, where the global time-scale (e.g. period of oscillation of a wing) can be much larger than the time-step, and for the high-Reynolds viscous flows, where the mesh refinement in boundary layers results in extremely small time-steps. A computation with explicit scheme requires in such cases big amount of computer time [4].

The goal of this work is to re-formulate the implicit MacCormack's finite-difference scheme given in [1] finite volume framework considering the arbitrary Lagrangian–Eulerian method. The resulting scheme combines advantages of the explicit method (second order accuracy, simplicity of implementation) with the power of implicit methods (unconditional stability).

2. Implicit MacCormack scheme for scalar problem

Before describing the final method for viscous flows in arbitrary Lagrangian–Eulerian formulation, a simple model initial value problem is considered:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \mu\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

with an initial condition $u(x,0) = u_0(x)$.

The explicit MacCormack scheme is realized in two steps:

Predictor:

$$\Delta u_i^n = -\frac{a\Delta t}{\Delta x}\left(u_{i+1}^n - u_i^n\right) + \frac{\mu\Delta t}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right), \tag{2}$$

$$u_i^{n+1/2} = u_i^n + \Delta u_i^n. \tag{3}$$

Corrector:

$$\Delta u_i^{n+1/2} = -\frac{a\Delta t}{\Delta x}\left(u_i^{n+1/2} - u_{i-1}^{n+1/2}\right) + \frac{\mu\Delta t}{\Delta x^2}\left(u_{i+1}^{n+1/2} - 2u_i^{n+1/2} + u_{i-1}^{n+1/2}\right), \tag{4}$$

$$u_i^{n+1} = \frac{1}{2}\left(u_i^n + u_i^{n+1/2} + \Delta u_i^{n+1/2}\right). \tag{5}$$

The explicit scheme is stable under CFL condition

$$\Delta t \leqslant \frac{1}{(|a|/\Delta x) + (2\mu/\Delta x)^2}. \tag{6}$$

The implicit scheme is obtained by replacing one-sided differences in convective terms (those with factor a) in predictor (2) by

$$\frac{a\Delta t}{\Delta x}\left(u_{i+1}^n - u_i^n\right) + \lambda\frac{\Delta t}{\Delta x}\left[\left(u_{i+1}^{n+1/2} - u_{i+1}^n\right) - \left(u_i^{n+1/2} - u_i^n\right)\right], \tag{7}$$

and the second order differences in viscous terms by

* Corresponding author.
E-mail addresses: Jiri.Furst@fs.cvut.cz (J. Fürst), Petr.Furmanek@fs.cvut.cz (P. Furmánek).

0045-7930/$ - see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.compfluid.2010.09.036

$$\frac{\mu\Delta t}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) + \lambda\frac{\Delta t}{\Delta x^2}\left[\left(u_{i+1}^{n+1/2} - u_{i+1}^n\right) - \left(u_i^{n+1/2} + u_i^n\right)\right], \tag{8}$$

and similarly for corrector (4)

$$\frac{a\Delta t}{\Delta x}\left(u_i^{n+1/2} - u_{i-1}^{n+1/2}\right) + \lambda\frac{\Delta t}{\Delta x}\left[\left(u_i^{n+1} - u_i^{n+1/2}\right) - \left(u_{i-1}^{n+1} - u_{i-1}^{n+1/2}\right)\right], \tag{9}$$

and

$$\frac{\mu\Delta t}{\Delta x^2}\left(u_{i+1}^{n+1/2} - 2u_i^{n+1/2} + u_{i-1}^{n+1/2}\right) + \lambda\frac{\Delta t}{\Delta x^2}\left[\left(u_i^{n+1} - u_i^{n+1/2}\right) - \left(u_{i-1}^{n+1} - u_{i-1}^{n+1}\right)\right]. \tag{10}$$

The final implicit finite-difference scheme is then (see [1]):

Predictor:

$$\Delta u_i^n = -\frac{a\Delta t}{\Delta x}\left(u_{i+1}^n - u_i^n\right) + \frac{\mu\Delta t}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right), \tag{11}$$

$$\left(1 + \frac{\lambda\Delta t}{\Delta x}\right)\delta u_i^{n+1/2} = \Delta u_i^n + \frac{\lambda\Delta t}{\Delta x}\delta u_{i+1}^{n+1/2}, \tag{12}$$

$$u_i^{n+1/2} = u_i^n + \delta u_i^{n+1/2}. \tag{13}$$

Corrector:

$$\begin{aligned}\Delta u_i^{n+1/2} = &-\frac{a\Delta t}{\Delta x}\left(u_i^{n+1/2} - u_{i-1}^{n+1/2}\right)\\ &+\frac{\mu\Delta t}{\Delta x^2}\left(u_{i+1}^{n+1/2} - 2u_i^{n+1/2} + u_{i-1}^{n+1/2}\right),\end{aligned} \tag{14}$$

$$\left(1 + \frac{\lambda\Delta t}{\Delta x}\right)\delta u_i^{n+1} = \Delta u_i^{n+1/2} + \frac{\lambda\Delta t}{\Delta x}\delta u_{i-1}^{n+1}, \tag{15}$$

$$u_i^{n+1} = \frac{1}{2}\left(u_i^n + u_i^{n+1/2} + \delta u_i^{n+1}\right). \tag{16}$$

Note that the predictor step is evaluated starting at the greatest index i using an appropriate boundary condition (e.g. $\delta u_N^{n+1/2} = 0$) and going to lowest index. The corrector step is evaluated in the similar manner starting with boundary condition for lowest index and going to greatest one.

The linear scheme is unconditionally stable provided that the explicit/implicit blending parameter λ is chosen such that

$$\lambda \geqslant \frac{1}{2}\max\left(|a| + \frac{2\mu}{\Delta x} - \frac{\Delta x}{\Delta t}, 0\right). \tag{17}$$

The conservativity of the scheme can be easily proven by summing the solution over an index interval $i \in [p, q]$, i.e.

$$\begin{aligned}\sum_{i=p}^{q} u_i^{n+1/2} - \sum_{i=p}^{q} u_i^n = \sum_{i=p}^{q}\delta u_i^{n+1/2} = \sum_{i=p}^{q}\left[\Delta u_i^{n+1/2} + \frac{\lambda\Delta t}{\Delta x}\left(\delta u_{i+1}^{n+1/2} - \delta u_i^{n+1/2}\right)\right]\\ = \frac{\Delta t}{\Delta x}\left[\left(au_p^n - \frac{\mu}{\Delta x}(u_p^n - u_{p-1}^n) - \lambda\delta u_p^{n+1/2}\right)\right.\\ \left.-\left(au_{q+1}^n - \frac{\mu}{\Delta x}(u_{q+1}^n - u_q^n) - \lambda\delta u_{q+1}^{n+1/2}\right)\right].\end{aligned} \tag{18}$$

Therefore, the total change of the solution over interval p, q in predictor is caused only by a fluxes through both interval ends (first and second parenthesis in last term). Similar is true for corrector.

One can easily see that all three steps in predictor can be evaluated together during one backward sweep through the mesh, i.e. it is not necessary to solve any system of linear equations. The same is valid for the corrector, which can be again realized by one forward sweep.

The choice of λ ensures that the scheme can switch to explicit one whenever the stability condition (6) is satisfied.

In order to get some insight on the properties of the implicit scheme we perform the Fourier analysis. Assume

$$u_i^n = C^n \exp(\mathscr{I}k\Delta x i), \tag{19}$$

where k is the (real) wave-number and $\mathscr{I} = \sqrt{-1}$. Plugging this formula into the scheme and using the computer algebra system Maxima the complex amplification factor C has been computed. The final form of C is quite complicated to analyze, nevertheless it is possible to expand the formula into power series of Δx and Δt up to third order:

$$\begin{aligned}\mathrm{Re}(C) = 1 - k^2\mu\Delta t + \frac{k^4\mu^2 - k^2a^2}{2}\Delta t^2\\ + \frac{\mu\frac{\Delta x^2}{\Delta t^2} + 6\mu\lambda\frac{\Delta x}{\Delta t} + 12\mu\lambda^2}{12}k^4\Delta t^3 + \cdots,\end{aligned} \tag{20}$$

$$\mathrm{Im}(C) = -ka\Delta t + k^3 a\mu\Delta t^2 + \frac{a\frac{\Delta x^2}{\Delta t^2} + 6a\lambda\frac{\Delta x}{\Delta t} + 6a\lambda^2}{6}k^3\Delta t^3 + \cdots \tag{21}$$

Plugging the harmonic solution $u(x,t) = c(t)\exp(\mathscr{I}kx)$ into the original equation one gets

$$u_t = -au_x + \mu u_{xx} = (-\mathscr{I}ak - \mu k^2)u = Ku, \tag{22}$$

and finally

$$\begin{aligned}u(x, t + \Delta t) &= \exp(K\Delta t)u(x,t)\\ &= \left(1 + K\Delta t + \frac{K^2\Delta t^2}{2} + \cdots\right)u(x,t).\end{aligned} \tag{23}$$

Direct calculation gives

$$\mathrm{Re}(\exp(K\Delta t)) = 1 - k^2\mu\Delta t + \frac{k^4\mu^2 - a^2k^2}{2}\Delta t^2 + \cdots, \tag{24}$$

$$\mathrm{Im}(\exp(K\Delta t)) = -ak\Delta t + ak^3\mu\Delta t^2 + \cdots \tag{25}$$

Comparing Eqs. (20), (21) and (24), (25) one see that the implicit scheme is second order accurate. Moreover, the implicit part introduces additional terms (with respect to modified equation of explicit MacCormack scheme) corresponding to

$$\Delta t^2\left[\mu\lambda\left(\lambda + \frac{\Delta x}{2\Delta t}\right)u_{xxxx} - a\lambda\left(\lambda + \frac{\Delta x}{\Delta t}\right)u_{xxx}\right]. \tag{26}$$

3. Numerical solution of simple model problem

Before going to full Euler or Navier–Stokes equations we present some preliminary results for the case of simple scalar linear initial value problem for Eq. (1). We assume a discontinuous initial condition $u(x,0) = 1$ for $x \in [0.25, 0.5]$ and 0 otherwise, and we compute the solution in time $t = 0.25$. Fig. 1 shows the numerical solution obtained with $\Delta x = 0.001$ obtained with different $a\Delta t/\Delta x$ ratios. The results show some oscillations arising at the discontinuities, nevertheless the scheme is still stable. Note that the oscillations in the vicinity of the discontinuities can be suppressed by using an appropriate artificial viscosity (e.g. TVD damping [5]).

4. The implicit scheme in finite volume formulation

The original finite-difference scheme is extended for 2D/3D Euler or Navier–Stokes equations using finite volume formulation. The so called arbitrary Lagrangian–Eulerian method is used for the case of moving meshes. Assume standard Navier–Stokes equations for compressible flows in 2D in the conservative form