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A moving mesh approach to an ice sheet model

D. Partridge *, M.J. Baines

School of Mathematics, Meteorology and Physics, University of Reading, Reading, United Kingdom

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ABSTRACT

A moving mesh approach to the numerical modelling of problems governed by nonlinear time-dependent partial differential equations (PDEs) is applied to the numerical modelling of glaciers driven by ice diffusion and accumulation/ablation. The primary focus of the paper is to demonstrate the numerics of the moving mesh approach applied to a standard parabolic PDE model in reproducing the main features of glacier flow, including tracking the moving boundary (snout). A secondary aim is to investigate waiting time conditions under which the snout moves.

1. Introduction/Background

In this paper we take a standard one-dimensional PDE model of a glacier and discretise it using a finite difference moving mesh method. The aim is to show that the numerical scheme can reproduce the features of the model and in particular handle the phenomenon of waiting time which is a known feature of glacier movement.

Computational studies of glaciers are particularly challenging to the modeller. Although ice sheet models are well-established, prediction of profiles and grounding movement are infeasible analytically and difficult to achieve numerically, see Payne and Vieli [5]. Ice moves in a similar manner to a viscous fluid, though with a very high viscosity approximately 10^{15} times that of water [2]. However, viscous theory cannot solely be used to describe flow, since glaciers are unique in experiencing basal sliding. This can be caused in two ways, via friction where the ice makes contact with the ground as it is flowing, or geothermal heat below the surface.

In order for glaciers to form they first need enough snow over the winter period to be able to survive through the summer, i.e. more accumulation of snow than is lost through melting and evaporation. This needs to be repeated over a number of successive years, and as more snow builds up, the weight increases and pressure compresses the firn (old snow) into ice. Once this ice is thick enough, gravity, amongst other forces, causes the ice to flow. This is a long, complex process which takes less time in regions where temperature changes quicker, such as the Alps and North America [2].

On a global scale, ice quantities vary considerably. At present glaciers make up around 2% of the Earths water, but during an ice age this vastly increases. Either way they have a large impact on the climate system, and are becoming increasingly affected by climate change. If all this ice melted into the oceans, there would be a sea level rise of around 70 m. We are interested in glaciers for more than just the climate change reasons, as they can have a large effect on the local terrain, causing events such as landslides and flash floods.

We discritise a standard PDE model of glacier movement in a moving frame of reference, on a moving mesh, using a local mass balance principle to define a velocity in order to move the mesh. We note that, as in other nonlinear diffusion problems, glaciers experience a waiting time before they begin to move. We suggest a mechanism whereby waiting ends and the snout moves. Finally, consideration is given to ways the model may be extended, and the impacts that these extensions may have on the results we have obtained, leading the way to potential further work to be undertaken on the problem.

One of the main concepts to take into consideration when modelling glaciers is the idea of mass balance, and where on the glacier mass is gained or lost. Generally, near the source of the glacier, the accumulation of snow is greater than the ablation (melting/evaporation), so the mass increases. Further away the ablation becomes greater than the accumulation, and the mass decreases. However ice can build up in the lower zone due to ice flow coming from the glacier's upper zone. The front-most end of the glacier is known as the snout, which rarely moves straight away; it waits until the velocity behind it is great enough to push it down the mountain. It is this feature which is of special interest.

* Corresponding author.
E-mail address: d.partridge@student.reading.ac.uk (D. Partridge).

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2. Model description

A standard PDE model for glaciers was proposed by Oerlemans [3] in 1984, with a flat bed occupying the region $x \in [0, b(t)]$ as shown in Fig. 1. Let $H(x,t)$ represent the thickness of the ice. At the ends of this domain we have the boundary conditions, $\frac{\partial H}{\partial x} = 0$ at the fixed point $x = 0$, and $H = 0$ at the moving boundary $x = b(t)$.

In one dimension the continuity equation for ice can be written as

$$\frac{\partial H}{\partial t} = -\frac{\partial (Hu)}{\partial x} + s(x), \tag{1}$$

where H is the ice thickness, $s(x) = s_a(x) - s_b(x)$, with s_a the accumulation rate of snow and s_b the basal melting rate.

2.1. Mass balance

An important property concerns the integral of the ice thickness over the whole domain (the volume), i.e.

$$\int_0^{b(t)} H(x,t)dx = \theta(t), \text{say.} \tag{2}$$

From (1), using Leibniz's integral rule, and applying the boundary conditions

$$\begin{aligned}\frac{d}{dt}\int_0^{b(t)} H(x,t)dx &= \int_0^{b(t)} \frac{\partial H}{\partial t} dx + H(b(t),t)\frac{db(t)}{dt} \\ &= -\int_0^{b(t)} \frac{\partial}{\partial x}[Hu]dx + \int_0^{b(t)} s(x)dx \\ &= -[Hu]_0^{b(t)} + \int_0^{b(t)} s(x)dx = \int_0^{b(t)} s(x)dx,\end{aligned} \tag{3}$$

the physical equivalent of which states that any change in the integral of ice thickness over the whole glacier, or equivalently any change in the ice volume, is due only to the snow term, which represents the net accumulation/ablation of snow over the whole glacier.

2.2. Model velocity

The model velocity u is defined as the mean depth integrated horizontal velocity, and assuming lamellar flow it is given by [9]

$$u = \frac{2AH}{n+2}\tau_{dx}^n, \tag{4}$$

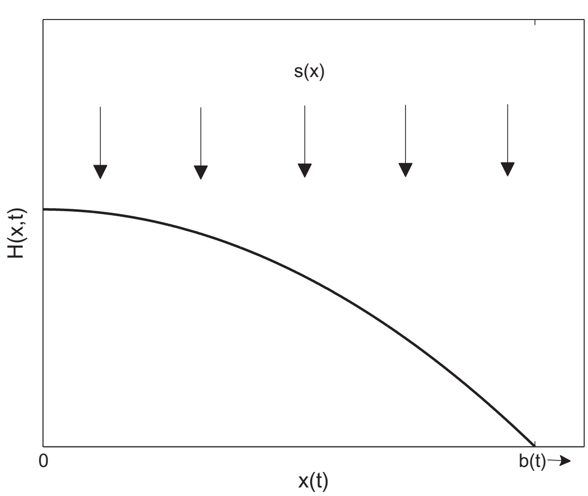

Fig. 1. One-dimensional domain.

with τ_{dx} the stress term, and parameters A and n taken from Glen's flow law, an established general law for steady state ice deformation [6]. From Van Der Veen [9] the driving stress is given by

$$\tau_{dx} = -\rho g H \frac{\partial h}{\partial x}, \tag{5}$$

with ρ the ice density, g representing gravity, and h equal to ice thickness plus the surface elevation. On a flat bed there is no surface elevation so we may put $h = H$. From (4) and (5) we get an equation for the depth integrated horizontal velocity

$$u = -\frac{2AH}{n+2}\rho^n g^n H^n \left(\frac{\partial H}{\partial x}\right)^n. \tag{6}$$

The parameters A, n, ρ and g are set as constant to simplify the model, giving

$$u = -cH^{n+1}\left(\frac{\partial H}{\partial x}\right)^n, \tag{7}$$

where c is a single positive constant parameter.

Expressing the velocity in this form may present problems when dealing with the boundary condition $H = 0$ at $x = b(t)$, apparently giving a zero velocity at the right-hand boundary and resulting in a glacier that will never move, which we know physically is not the case. However it is perfectly possible for u to be non-zero as long as $H^{n+1}H_x^n$ is finite, which requires H_x to be infinite.

For the most part though we are not concerned with physical values for the variables, but more with the numerical behaviour of the moving mesh approach, hence x and H are non-dimensionalised, with $\tilde{x} = 10^{-6}x$ and $\widetilde{H} = 10^{-3}H$. We drop the tilde notation for convenience. From Roberts [7] we set $c = 0.000022765$ in standard SI units, and $n = 3$. Substituting the velocity into Eq. (1) we get the model equation

$$\frac{\partial H}{\partial t} = c\frac{\partial}{\partial x}\left[H^5 H_x^3\right] + s(x), \tag{8}$$

which incorporates nonlinear diffusion and a source term. In this paper we set $s_b(x) \equiv 0$, making $s(x) = s_a(x)$, although the non-zero basal melting case is considered in [4].

3. Snout behaviour

From (7), with $n = 3$ we derive the useful form

$$u = -c(H^{4/3}H_x)^3 = -\frac{27}{343}c[(H^{7/3})_x]^3. \tag{9}$$

When expressing the velocity in this manner it is interesting to substitute an expression for H that has the right general shape and satisfies the boundary conditions, i.e.

$$H = (1 - x^2)^\alpha \tag{10}$$

where $\alpha > 0$, for which

$$\begin{aligned} H^{\frac{7}{3}} &= (1 - x^2)^{\frac{7\alpha}{3}} \\ (H^{7/3})_x &= -2x.\frac{7\alpha}{3}(1 - x^2)^{\frac{7\alpha}{3}-1}. \end{aligned} \tag{11}$$

The velocity (9) then has some interesting properties as $x \to 1$, depending on the value of α.

$$\text{Case 1}: \frac{7\alpha}{3} > 1, \Rightarrow (H^{7/3})_x \text{is zero} \tag{12}$$

$$\text{Case 2}: \frac{7\alpha}{3} < 1, \Rightarrow (H^{7/3})_x \text{is infinite} \tag{13}$$

$$\text{Case 3}: \frac{7\alpha}{3} = 1, \Rightarrow (H^{7/3})_x \text{is finite} \tag{14}$$