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A mimetic mass, momentum and energy conserving discretization for the shallow water equations

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ABSTRACT

A spatial semi-discretization is developed for the two-dimensional depth-averaged shallow water equations on a non-equidistant structured and staggered grid. The vector identities required for energy conservation in the continuous case are identified. Discrete analogues are developed, which lead to a finite-volume semi-discretisation which conserves mass, momentum, and energy simultaneously. The key to discrete energy conservation for the shallow water equations is to numerically distinguish storage of momentum from advective transport of momentum. Simulation of a large-amplitude wave in a basin confirms the conservative properties of the new scheme, and demonstrates the enhanced robustness resulting from the compatibility of continuity and momentum equations. The scheme can be used as a building block for constructing fully conservative curvilinear, higher order, variable density, and non-hydrostatic discretizations.

1. Introduction

When systems of partial differential equations are discretized, properties of the continuous equations are often inadvertently sacrificed. One pertinent example is that many finite-difference and finite-volume discretizations of the shallow water equations are not energy conserving (e.g. [1–3]), even in absence of friction and energy fluxes through domain boundaries. Other examples are schemes often used for geophysical applications, which conserve mass, energy, potential enstrophy, but not momentum (e.g. [4]), or a recently developed curl-preserving discretisation of the shallow water equations [5] which does not conserve energy. In general, one cannot avoid compromises in the discretisation of the continuous system; which properties are to be preserved depends on (1) the dominating physics of the problem under consideration, and (2) a knowledge of the robustness and accuracy of the discretisation.

The wider aim of this work is to develop numerical schemes suitable for simulating thee-dimensional (3-D) turbulent flows with a free surface. In 3-D turbulence, energy cascades from large scales to ever smaller scales until it is converted into heat by viscosity. Therefore, it is important to design numerical schemes which preserve the subtle balances of energy transfer between scales. In this paper, we take a first step in this direction by deriving a scheme for the shallow water equations which conserves mass, momentum, and energy simultaneously. Whilst mass and momentum conservation is straightforward to enforce using a finite-volume discretization, energy conservation is much harder to achieve. Indeed, energy conservation cannot be enforced directly, and requires a compatible discretization of the continuity and momentum equations. From a numerical point of view, energy conservation is a desirable property for the numerical stability and robustness of the discretization, because the velocities will remain bounded.

It has long been recognized that energy conservation is an important issue in numerical simulation. Fully conservative schemes for the solution of the incompressible Navier–Stokes equations on staggered Cartesian grids (in particular in the context of direct numerical and large-eddy simulation) have been derived for finite difference [6], finite volume [7] and compact difference methods [8]. Recently, a collocated scheme was developed for the incompressible Navier–Stokes equations which is fully conservative [9]. However, none of these schemes can be applied to the shallow water equations, because they were not designed for time-dependent cell volumes. In this paper, we seek a momentum- and energy-preserving discretization of the two-dimensional (2-D) shallow water equations on a Eulerian staggered grid.

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The 2-D depth-averaged shallow water equations in the absence of friction in conservative form are given by

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0, \tag{1}$$

$$\frac{\partial h\mathbf{u}}{\partial t} + \nabla \cdot (h\mathbf{u}\mathbf{u}) + gh\nabla\zeta = 0. \tag{2}$$

Here h is the water depth, $\mathbf{u} = [u, v]^*$ is the depth-averaged fluid velocity, ζ is the free surface elevation relative to the water interface at rest and g is the gravitational acceleration. The bed is located at $z = -d(x,y)$ and the water depth is $h = \zeta + d$.

We define a scalar product as

$$\langle \mathbf{u}, \mathbf{v} \rangle \equiv \int_\Omega \mathbf{u} \cdot \mathbf{v} d\Omega, \tag{3}$$

where Ω is the fluid domain. Assuming that the fluxes on the domain boundary $\partial\Omega$ vanish, the total mass $\langle 1, h\rangle$, momentum $\langle \mathbf{1}, h\mathbf{u}\rangle$ and twice the energy $\langle \mathbf{u}, h\mathbf{u}\rangle$ evolve according to

$$\frac{d}{dt}\langle 1, h\rangle = 0, \tag{4}$$

$$\frac{d}{dt}\langle \mathbf{1}, h\mathbf{u}\rangle = \langle \mathbf{1}, gh\nabla d\rangle, \tag{5}$$

$$\frac{d}{dt}(\langle \mathbf{u}, h\mathbf{u}\rangle + g\langle \zeta, \zeta\rangle) = 0. \tag{6}$$

Hence, in the situation that the bed is flat, mass, momentum and energy are conserved. Using a finite-volume discretization, mass and momentum conservation can be enforced explicitly. However, energy conservation can only be ensured by choosing the discretization such that two important integral relations pertaining to advection and pressure are satisfied. These relations will be derived below.

The rate of change of twice the total kinetic energy $\langle \mathbf{u}, h\mathbf{u}\rangle$ is defined by

$$\frac{d}{dt}\langle \mathbf{u}, h\mathbf{u}\rangle = \left\langle \frac{\partial h\mathbf{u}}{\partial t}, \mathbf{u} \right\rangle + \left\langle \mathbf{u}, \frac{\partial h\mathbf{u}}{\partial t} \right\rangle - \left\langle \mathbf{u}\cdot\mathbf{u}, \frac{\partial h}{\partial t} \right\rangle. \tag{7}$$

Substituting Eq. (2) into the expression above, and assuming that boundary fluxes vanish or cancel, results in

$$\frac{d}{dt}\langle \mathbf{u}, h\mathbf{u}\rangle = -\langle \mathscr{A}\mathbf{u}, \mathbf{u}\rangle - \langle \mathbf{u}, \mathscr{A}\mathbf{u}\rangle - \left\langle \mathbf{u}, \frac{\partial h}{\partial t}\mathbf{u} \right\rangle - \langle gh\nabla\zeta, \mathbf{u}\rangle - \langle \mathbf{u}, gh\nabla\zeta\rangle. \tag{8}$$

Here, $\mathscr{A}$ is the advective operator, defined as $\mathscr{A}\mathbf{v} = \nabla\cdot(h\mathbf{u}\mathbf{v})$. Using partial integration and making use of Eq. (1), it follows that

$$\langle \mathscr{A}\mathbf{u}, \mathbf{v}\rangle = -\langle \mathbf{u}, \mathscr{A}\mathbf{v}\rangle + \langle \mathbf{u}\cdot\mathbf{v}, \nabla\cdot(h\mathbf{u})\rangle, \tag{9}$$

$$\langle \nabla\zeta, h\mathbf{u}\rangle + \langle h\mathbf{u}, \nabla\zeta\rangle = \frac{d}{dt}\langle \zeta, \zeta\rangle. \tag{10}$$

Substitution of the identities (9) and (10) into Eq. (8) results in Eq. (6).

To achieve energy conservation for the numerical method, discrete analogues of (9) and (10) need to be satisfied. This approach falls within the category of mimetic discretizations [10,11]; see [7,12] for derivations using this approach for the incompressible Navier–Stokes equations.

2. A mass, momentum and energy conserving discretization

Recently, a shift transformation framework was developed, which allows one to construct mass, momentum, and energy conserving discretizations on staggered grids by making use of a fully conservative discretization of the advection–diffusion equation for a cell-centered scalar [12]. The advantage of this approach is that it is straightforward to a construct a variance conserving scheme for a cell-centered variable, whereas fully conservative equations on a staggered mesh are much harder to derive.

The advection equation for a scalar is given by

$$\frac{\partial hc}{\partial t} + \nabla\cdot(h\mathbf{u}c) = 0, \tag{11}$$

and a generic finite-volume discretization of Eqs. (1) and (11) is

$$\frac{d\mathbf{V}(t)}{dt} + \mathscr{D}\mathbf{u} = 0, \tag{12}$$

$$\frac{dM(t)\mathbf{c}(t)}{dt} + \mathscr{A}(t)\mathbf{c} = 0. \tag{13}$$

Here, $\mathbf{u}$ and $\mathbf{c}$ are vectors containing the velocity and concentration, respectively. The cell volume vector $\mathbf{V}$ is defined as $\mathbf{V} = \Omega\mathbf{h}$, where Ω is a diagonal matrix containing the cell areas. The matrices $\mathscr{D}$ and $\mathscr{A}$ are the discrete divergence and advection operators, respectively; M is a diagonal mass matrix with $\mathrm{diag}(M) = \mathbf{V}$. The primary difference between Eqs. (12) and (13) and an incompressible formulation is that the mass matrix M is now time-dependent and the depth-averaged velocity field is not divergence free.

The time rate of change of the total variance $\mathbf{c}^*M\mathbf{c}$ is

$$\frac{d}{dt}\mathbf{c}^*M\mathbf{c} = \left(\frac{dM\mathbf{c}}{dt}\right)^*\mathbf{c} + \mathbf{c}^*\frac{dM\mathbf{c}}{dt} - \mathbf{c}^*\frac{dM}{dt}\mathbf{c} = -\mathbf{c}^*\left(\mathscr{A} + \mathscr{A}^* + \frac{dM}{dt}\right)\mathbf{c}, \tag{14}$$

which is a discrete equivalent of Eq. (8) for a scalar. Requiring that the total variance be conserved imposes the following constraints on the discrete advective operator:

$$\begin{cases} \mathrm{diag}(\mathscr{A}) = -\frac{1}{2}\mathrm{diag}\left(\frac{dM}{dt}\right) = \frac{1}{2}\mathscr{D}\mathbf{u} & \text{(diagonal elements.)}, \\ \mathscr{A} = -\mathscr{A}^* & \text{(off-diagonal elements.)}. \end{cases} \tag{15}$$

Hence, the off-diagonal terms need to be skew-symmetric, and the divergence of the volume flux features on the diagonal of the operator. In the case of the incompressible Navier–Stokes equations, the field is divergence free, which results in a strictly skew-symmetric advective operator.

Although the argument above is valid for both structured and unstructured grids, we specialize to a non-equidistant rectangular staggered C-grid [13,14] (Fig. 1). The centers of the north and east cell faces are located at x_i, and y_j, respectively. The cell center is located at $(x_{i-1/2}, y_{j-1/2})$. Fractional indices denote a non-weighted interpolation $X_{i+1/2} \equiv \frac{1}{2}(X_i + X_{i+1})$, where X is an arbitrary field variable. The discrete variables are defined as $u_{i,j} = u(x_i, y_{j-1/2})$, $v_{i,j} = v(x_{i-1/2}, y_j)$, and $h_{i,j} = h(x_{i-1/2}, y_{j-1/2})$.

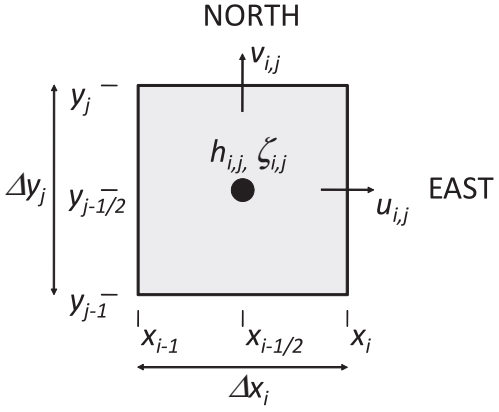

Fig. 1. Definition sketch of the grid and the dependent variables.