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The method of fundamental solutions for oscillatory and porous buoyant flows

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ABSTRACT

Accurate solutions of oscillatory Stokes flows in convection and convective flows in porous media are studied using the method of fundamental solutions (MFS). In the solution procedure, the flows are represented by a series of fundamental solutions where the intensities of these sources are determined by the collocation on the boundary data. The fundamental solutions are derived by transforming the governing equation into the product of harmonic and Helmholtz-type operators, which can be classified into three types depending on the oscillatory frequencies of temperature field. All the velocities, the pressure, and the stresses corresponding to the fundamental solutions are expressed explicitly in tensor forms for all the three cases. Three numerical examples were carried out to validate the proposed fundamental solutions and numerical schemes. Then, the method was also applied to study exterior flows around a sphere. In these studies, we derived the MFS formulas of drag forces. Numerical results were compared accurately with the analytical solutions, indicating the ability of the MFS for obtaining accurate solutions for problems with smooth boundary data. This study can also be treated as a preliminary research for nonlinear convective thermal flows if the particular solutions of the operators can be supplied, which are currently under investigations.

1. Introduction

Recently, buoyant flows induced by time-periodic boundary temperature variations have become a subject of considerable interests. Examples include the buoyant flow in a cavity induced by sinusoidal sidewall temperature variations [1] and others [2–5]. When the acceleration forces dominate the nonlinear inertial forces and the amplitude of oscillatory temperature input is small enough, the hydrodynamic and energy equations can be linearized. In this situation, the hydrodynamic and energy equations become Brinkman and modified Helmholtz equations respectively if harmonic vibrations in time are further assumed. On the other hand, when boundary-type numerical methods [6] are applied to study fluid dynamic problems, the linearized governing equations are usually the first relevant subjects [7–10]. Therefore, in this paper we are going to develope the method of fundamental solutions (MFS) for oscillatory buoyant flows.

On the other hand, the transport phenomena in porous media arise in many diverse fields of science and engineering, such as civil, mechanical, chemical, and petroleum engineering. Thus, the analysis of transport phenomena in porous media is of great importance in science and engineering. Since the original study of Darcy [11], the transport phenomena in porous media had been studied extensively over years. A large amount of these studies are based on the Brinkman extended Darcy's model [12]. When there are heat sources in porous media, the governing equations of hydrodynamics and energy become Brinkman and Laplace equations, respectively. In this paper, we will also apply the MFS to this subject.

In the last few decades, there are increasing interests in the developments of the MFS for various engineering and scientific problems. The MFS is a boundary-type numerical method which was first proposed by Kupradze and Aleksidze [13]. Its mathematical foundations were established by Mathon and Johnston [14] and Bogomolny [15]. Then, the MFS had been successfully applied to the elliptic boundary value problems [16], the scattering and radiation problems [17], the evaluation of eigenvalues [18], and the diffusion problems [19]. In this paper, we developed the MFS formulations for oscillatory and porous buoyant flows. The hydrodynamic equation for both problems is Brinkman equation but the energy equations are modified Helmholtz and Laplace equations respectively. In the study we assume the temperature field is unaffected by the fluid motion. In other words we are going to study the particular solutions of Brinkman equation when the buoyant temperature terms are approximated by the fundamental solutions of energy equation (modified Helmholtz/Laplace equation). Karageorghis and Smyrlis [20] had made a similar study for thermoelasticity.

The MFS has also been applied to solve many problems of fluid flows. Tsai and Young [21] applied the MFS to solve

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three-dimensional Stokes problems. In that study they solved the vorticity transport equation and the velocity Poisson equation separately by using the MFS based on the modified Helmholtz and Laplace fundamental solutions, respectively. Studies of the MFS by directly using the fundamental solution of hydrodynamic equation are originated from Alves and Silvestre [22] and Tsai et al. [23] for interior and exterior Stokes flow problems, respectively. On the other hand, Tsai [10] applied the MFS to study transport phenomena in porous media by directly using the Brinkman fundamental solutions. In this study, we extended the MFS for the case of buoyant flows in porous media.

Meanwhile Tsai et al. [24] used the unsteady fundamental solution to solve unsteady Stokes problems. However, the accuracy of that study was sensitive with respect to the locations of sources. Alternatively, Pozrikidis [7,8] and Shatz [9] derived the singularity method for oscillating Stokes problems. In their studies all kinds of sources are adopted to construct the solutions. In the recent theories of the MFS [15,22], it had been shown that simply putting the fundamental solutions on an artificial boundary outside (or inside) the computational domain is enough to form a dense space for all the solutions of interior (or exterior) problems. Therefore we also investigate the oscillatory buoyant flows based on this concept.

A brief outline of the paper is as follows. In Section 2, we introduce the governing equations. Then, we derive the fundamental solutions required for the MFS formulations in Section 3. In our derivations, we first transform the governing equation to the product of harmonic and Helmholtz-type operators using the Hörmander operator decomposition technique [25]. Depending on the oscillatory frequencies of temperature field the product operator is classified to three types, whose fundamental solutions can be found in literature [26]. Then, the MFS formulation is introduced in Section 4. The drag forces of exterior problems are studied in Section 5. The numerical results are then presented in Section 6 and the conclusions are summarized in Section 7.

2. Governing equations

For a linearized unsteady buoyant Stokes flow with the Oberbeck–Boussinesq assumption [27,28], the mass, momentum, and energy conservation equations are given, respectively, by Pozrikidis et al. [7,29]:

$$\begin{cases} \nabla\cdot\bar{\mathbf{u}}=0 \\ \rho\frac{\partial\bar{\mathbf{u}}}{\partial t}=-\nabla\bar{p}+\mu\nabla^2\bar{\mathbf{u}}-\rho\beta\overline{T}\mathbf{g} \quad \text{in } \Omega \\ \frac{\partial\overline{T}}{\partial t}=k_T\nabla^2\overline{T} \end{cases} \tag{1}$$

where Ω is the study domain, $\bar{\mathbf{u}}=(\bar{u}_1,\bar{u}_2,\bar{u}_3)$ (or $\bar{\mathbf{u}}=(\bar{u}_1,\bar{u}_2)$) is the velocity vector in 3D (or 2D), $\bar{p}$ is the pressure, $\overline{T}$ is the temperature difference with respect to some proper reference temperature, $\mathbf{g}=(0,0,-g)$ (or $\mathbf{g}=(0,-g)$) is the gravity vector in 3D (or 2D), μ is the viscosity, ∇ is the gradient operator as usual, k_T is the thermal conductivity, and β is the coefficient of thermal expansion. For convenience, we assume $\bar{\mathbf{u}},\bar{p}$, and $\overline{T}$ are harmonic function in time as follows:

$$\begin{aligned} &\bar{\mathbf{u}}(\mathbf{x},t)=\mathbf{u}(\mathbf{x})e^{\mathbf{i}\omega t} \\ &\bar{p}(\mathbf{x},t)=p(\mathbf{x})e^{\mathbf{i}\omega t} \\ &\overline{T}(\mathbf{x},t)=T(\mathbf{x})e^{\mathbf{i}\omega t} \end{aligned} \tag{2}$$

where $\mathbf{x}=(x_1,x_2,x_3)$ (or $\mathbf{x}=(x_1,x_2)$) is the spatial coordinate in 3D (or 2D), $\mathbf{i}=\sqrt{-1}$ is the complex unit and ω is the frequency. Then Eq. (1) becomes

$$\begin{cases} \nabla\cdot\mathbf{u}=0 \\ \mathbf{i}\omega\rho\mathbf{u}=-\nabla p+\mu\nabla^2\mathbf{u}-\rho\beta T\mathbf{g} \quad \text{in } \Omega \\ \mathbf{i}\omega T=k\nabla^2 T \end{cases} \tag{3}$$

If we further assume $\lambda^2=\mathbf{i}\omega\rho\mu$, $\sigma^2=\frac{\mathbf{i}\omega}{k}$, $\varepsilon=1$, and $\boldsymbol{\alpha}=-\rho\beta\mathbf{g}$ with $|\boldsymbol{\alpha}|=\alpha$, we have

$$\begin{cases} \nabla\cdot\mathbf{u}=0 \\ -\nabla p+\mu\nabla^2\mathbf{u}-\lambda^2\mu\mathbf{u}+\boldsymbol{\alpha}T=0 \\ k_T\nabla^2T-\varepsilon\sigma^2k_TT=0 \end{cases} \tag{4}$$

In Eq. (4), we notice that the hydrodynamic and energy equations are oscillatory Stokes (Brinkman) and modified Helmholtz equations, respectively.

Next, we consider a slow Brinkman-extended Darcy's flow in steady state with heat convection satisfying Oberbeck–Boussinesq assumption, the mass, momentum, and energy conservation equations are given, respectively, by Brinkman et al. [12,30]:

$$\begin{cases} \nabla\cdot\mathbf{u}=0 \\ -\nabla p+\mu\nabla^2\mathbf{u}-\frac{\mu}{\kappa}\mathbf{u}-\rho\beta T\mathbf{g}=0 \quad \text{in } \Omega \\ k_T\nabla^2\overline{T}=0 \end{cases} \tag{5}$$

where κ is the permeability coefficient and the other variables are similar to those in the previous case. Eq. (4) can be also obtained by assuming $\varepsilon=0$, $\lambda^2=\frac{1}{\kappa}$ and $\boldsymbol{\alpha}=-\rho\beta\mathbf{g}$. In this situation the hydrodynamic and energy equations are Brinkman and Laplace equations, respectively.

To form a well-posed problem in addition to Eq. (4), proper boundary conditions should be imposed:

$$\begin{cases} \mathbf{u}=\hat{\mathbf{u}} \quad \text{on } \Gamma_1^u \\ \mathbf{t}=\hat{\mathbf{t}} \quad \text{on } \Gamma_2^u \\ T=\widehat{T} \quad \text{on } \Gamma_1^T \\ \frac{\partial T}{\partial\mathbf{n}}=\widehat{T}_{\mathbf{n}} \quad \text{on } \Gamma_2^T \end{cases} \tag{6}$$

where $\hat{\mathbf{u}}$, $\hat{\mathbf{t}}$, $\widehat{T}$, and $\widehat{T}_{\mathbf{n}}$ are prescribed boundary data, $\mathbf{n}=(n_1,n_2,n_3)$ is the outward normal vector, and $\Gamma=\Gamma_1^u+\Gamma_2^u=\Gamma_1^T+\Gamma_2^T$ is the boundary of the computational domain Ω. It is desirable to define the traction boundary condition $\mathbf{t}=(t_1,t_2,t_3)$ by

$$t_i=\sigma_{ij}n_j \tag{7}$$

with σ_{ij} is the stress tensor given by

$$\sigma_{ij}=-p+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \tag{8}$$

The main task of present study is to develop the MFS formulations for Eqs. (4) and (6). In other words, we are going to find the particular solutions of $\mathbf{u}$ and p, governed by the continuity and momentum equation in Eq. (4) when the temperature T are approximated by the modified Helmholtz and Laplace fundamental solutions.

3. The fundamental solutions

In the derivations, the following notation convections are utilized. The index γ varies from 1 to 3 and 1 to 4 for 2D and 3D, respectively. On the other hand, the indices $\{i,j,k\}$ take their values from 1 to 2 and 1 to 3 for 2D and 3D, respectively.

Then, we introduce the fundamental solutions required in the MFS formulations. For two dimensions, Eq. (4) can also be rewritten in matrix form as

$$\widetilde{L}\begin{pmatrix} u_1 \\ u_2 \\ T \\ p \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{9}$$

where $\widetilde{L}$ is a matrix of differential operators defined by