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Dynamic system uncertainty propagation using polynomial chaos

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Abstract The classic polynomial chaos method (PCM), characterized as an intrusive methodology, has been applied to uncertainty propagation (UP) in many dynamic systems. However, the intrusive polynomial chaos method (IPCM) requires tedious modification of the governing equations, which might introduce errors and can be impractical. Alternative to IPCM, the non-intrusive polynomial chaos method (NIPCM) that avoids such modifications has been developed. In spite of the frequent application to dynamic problems, almost all the existing works about NIPCM for dynamic UP fail to elaborate the implementation process in a straightforward way, which is important to readers who are unfamiliar with the mathematics of the polynomial chaos theory. Meanwhile, very few works have compared NIPCM to IPCM in terms of their merits and applicability. Therefore, the mathematic procedure of dynamic UP via both methods considering parametric and initial condition uncertainties are comparatively discussed and studied in the present paper. Comparison of accuracy and efficiency in statistic moment estimation is made by applying the two methods to several dynamic UP problems. The relative merits of both approaches are discussed and summarized. The detailed description and insights gained with the two methods through this work are expected to be helpful to engineering designers in solving dynamic UP problems.

1. Introduction

Dynamic system modeled as a set of ordinary differential equations (ODEs) widely exists in practical applications, such as missile trajectory design, satellite orbit planning, and Mars probe landing task, where the guarantee of the accuracy of the designed state trajectory is a major concern. It is oftentimes unavoidable that uncertainty is present in the initial conditions and system parameters, which may introduce variation of the designed trajectory and consequently paralyze the system. Therefore, it is necessary to study the impact of the uncertainty on the state trajectory, i.e. dynamic uncertainty propagation (UP). UP for dynamic systems has received much attention in recent years. State-of-the-art algorithms for dynamic UP often assume that the system can be modeled as a linear Gaussian process. However, as demonstrated in Ref.[1], propagation

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of the state uncertainty fails to remain Gaussian for long integration time in the presence of highly nonlinear dynamics. To address this issue, there are many popular nonlinear methods including Monte-Carlo (MC),[2] Markov chain MC (MCMC),[3] Gaussian mixtures,[4] unscented Kalman filtering (UKC),[5] and Fokker–Planck–Kolmogorov framework.[6] An alternative approach to the dynamic UP is the polynomial chaos method (PCM, also named as stochastic finite element),[7–10] which is based on the original theory of Wiener on homogeneous chaos.[11] One notable advantage of the PCM is that the analytical expression of the state uncertainty can be obtained through expanding the uncertain quantities in terms of a weighted summation of certain prescribed random orthogonal polynomial basis functions. Generally, the PCM used at present is built on the Wiener–Askey polynomial chaos.[9] The original PCM is characterized as an intrusive methodology with which the system governing equations have to be extensively altered. The intrusive polynomial chaos method (IPCM) has been applied to dynamic UP in many problems. As the extensions of Wiener's polynomial chaos and Galerkin projections, a new method employing the Wiener–Askey polynomial chaos for solving stochastic differential equations was proposed.[9] Built upon IPCM, a stochastic spectral method to model uncertainty and its propagation in simulations of incompressible flows was developed.[10] With IPCM, a computational framework employing was presented to analyze the evolution of the uncertainty in state trajectory of a hypersonic air vehicle considering initial condition uncertainty and parameter uncertainty, in which the produced results achieve great agreements to those of MC at much more efficient computational cost.[12] IPCM was also applied to dynamic UP in solving trajectory optimization problems.[13] However, in order to apply the intrusive method, the existing codes have to be materially modified, which is very tedious at the risk of bringing about errors, hence is not preferable especially for some industrial codes that have been well validated. In some cases (e.g. the trajectory simulation model established in the MATLAB/Simulink platform, which has been modularized and validated based on physical experiments), altering the simulation codes is practically impossible, not to mention the coding error that may be inadvertently created. As an alternative approach to the classic intrusive method, the non-intrusive polynomial chaos method (NIPCM) has been proposed in Refs.[14–16], in which the whole governing equations are considered as a black-box-type function so that it is kept intact with no modifications. This form of polynomial chaos has been extensively applied to mechanical or structure mechanics problems exhibiting impressive accuracy and efficiency. Literature has also seen lots of applications of NIPCM to dynamic problems.[17–21] However, very few early works have compared NIPCM to IPCM in terms of their merits and applicability. It is the interest of the present paper to comparatively discuss the mathematical procedures of both polynomial chaos forms (IPCM and NIPCM) for dynamic UP. The two methods are then applied to dynamic UP for statistic moment estimation, of which the relative accuracy and efficiency are compared and discussed. Meanwhile, the results of the MC method are further employed as the reference to benchmark both approaches.

The rest of this paper is organized as follows. In Section 2, the procedures for the application of NIPCM as well as IPCM to dynamic UP are described in detail. In Section 3, NIPCM and IPCM are applied to two numerical examples and a gliding trajectory problem for dynamic UP. The produced results of both methods are compared and further verified against the MC method. Remarks and conclusion will be made in the final section.

2. Computation scheme

As mentioned in the introduction, the PCM employed in this work is built on the Wiener–Askey polynomial chaos.[9] Generally, a dynamic system is modeled as a set of ODEs with dimension q.

$$\begin{cases} \dfrac{\mathrm{d}y_i(t)}{\mathrm{d}t} = f_i(y, a, t) & (i = 1,\ 2,\ \ldots,\ q) \\ y(t_0) = y_0 & t \in [t_0, t_f] \end{cases} \tag{1}$$

where $\boldsymbol{y} = [y_1\, y_2 \ldots y_q]$ represents the state variables, and a is a constant parameter, y_0 are the initial state values, and the integration time span is $t \subseteq [t_0,\ t_f]$, t_0 and t_f are the initial and terminal times. For convenience, when introducing the procedures of IPCM and NIPCM for dynamic UP, the dynamic system with only one state variable ($q = 1$) is considered for convenience because situations with $q > 1$ can be easily expanded from the case with $q = 1$. If no uncertainty exists, any numerical integration approaches such as the commonly used Runge–Kutta method can be utilized to solve the deterministic ODEs (Eq. (1)). With the consideration of uncertainties, the original governing ODEs become stochastic ODEs. Actually, the dynamic UP here is essentially the process of solving the stochastic ODEs. Without loss of generality, two types of uncertainties are considered in this work, uncertainty from the initial condition $y(t_0)$ and uncertainty from the parameter a. With the consideration of these uncertainties, $y(t)$ becomes stochastic, which can be quantified by probabilistic measures such as mean and variance. Although the procedure of using IPCM for dynamic UP has been outlined in the above researches during the introduction part, it is described with more details here for the sake of clear comparison between the two forms of PCM. Taking the dynamic system in Eq. (1) with $q = 1$ as an example, the processes of applying IPCM and NIPCM to dynamic UP are elaborated as follows, respectively.

2.1. IPCM for dynamic UP

A step-by-step description of using IPCM for UP in dynamic system with the consideration of both uncertainties in $y(t_0)$ and a is given below. Readers can refer to Refs.[9,10,12] for more details.

Step 1. Since a and $y(t_0)$ are random, $y(t)$ is random. Based on the polynomial chaos theory, both the state $y(t)$ and the stochastic parameter a can be represented as polynomial chaos expansion (PCE) models as

$$\begin{cases} y(t) = \displaystyle\sum_{i=0}^{\infty} y_i(t)\Phi_i(\boldsymbol{\xi}(\theta)) \\ a = \displaystyle\sum_{i=0}^{\infty} a_i\Phi_i(\boldsymbol{\xi}(\theta)) \end{cases} \tag{2}$$

where $\boldsymbol{\xi}(\theta) = [\xi_1(\theta)\ \xi_2(\theta) \ldots \xi_d(\theta)]$ is a standard random vector of dimension d, θ is a parameter indicating that the quantities involved are random variables defined over a space of random