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Generalized symmetry classifications, integrable properties and exact solutions to the general nonlinear diffusion equations☆

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ABSTRACT

In this paper, the combination of generalized symmetry classification and recursion operator method is developed for dealing with nonlinear diffusion equations (NLDEs). Through the combination approach, all of the second and third-order generalized symmetries of the general nonlinear diffusion equation are obtained. As its special case, the recursion operators of the nonlinear heat conduction equation are constructed, and the integrable properties of the nonlinear equations are considered. Furthermore, the exact and explicit solutions generated from the generalized symmetries are investigated.

1. Introduction

In this paper, we consider the general nonlinear diffusion equation (NLDE) as follows:

$$u_t = \alpha u^p u_{xx} + \beta u^q u_x^2, \tag{1.1}$$

where $u = u(x, t)$ denotes the unknown function of the space variable x and time t, the parameters α, β, p and q are all arbitrary constants. In general, we assume that $p \neq 0$ and $\alpha\beta \neq 0$ throughout this paper. This assumption implies that Eq. (1.1) does not include the Burgers' equation (BE) as its special case, and the following discussion will show that it is far more complicated than the latter [1–3].

Eq. (1.1) includes a lot of important nonlinear equations as its special cases. For example, if $\alpha = 1$, $\beta = -2$, $p = -2$ and $q = -3$, then Eq. (1.1) becomes the following important nonlinear hear conduction equation:

$$u_t = \left(\frac{u_x}{u^2}\right)_x, \quad u = u(x, t) \neq 0. \tag{1.2}$$

Such nonlinear diffusion equations are of great importance in mathematical physics, integrable system, fluid mechanics and nonlinear theory, etc. In practice, many nonlinear wave models can be depicted by such equations [4,5]. In [4], Qu considered some new exact solutions to Eq. (1.2) and other nonlinear wave equations by a generalized conditional symmetry method. Barone

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et al. [5] studied Eq. (1.2) also by constructing the corresponding Dirichlet-to-Neumann map. Recently, we investigated the complete group classifications, generalized symmetries, Bäcklund transformations (BTs) and exact solutions to some nonlinear partial differential equations (NLPDEs) by the symmetry analysis method [1–3,6–12].

The Lie symmetry analysis is a powerful and systematic method for tackling symmetries, exact solutions and integrable properties of partial differential equations [1–4,6–23], and the formal recursion operator method is in the core of the symmetry approach [20]. In [2,3], the generalized symmetries and Bäcklund transformations (BTs) of the Burgers' equations are obtained by the recursion operator method. In [6–12], we studied the Lie group classifications, conservation laws (CLs) and exact solutions to some nonlinear PDEs, such group classifications are complete with respect to all of the parameters and given functions of the equations. However, these Lie group classifications of the equations are complete in the sense of geometric symmetry after all, so the symmetries are point symmetries actually. In the current paper, we shall develop the combination of generalized symmetry classification and the recursion operator method for dealing with the nonlinear diffusion equations, the complete generalized symmetry classifications, integrable properties and exact solutions to the equations are investigated for the first time in the literature.

As is well known, admitting infinite number of symmetries and conservation laws are intrinsic properties shared by many integrable systems. The generalized symmetry is the generalization of Lie point symmetry, and it should include the Lie point symmetry as its special case by nature. In [13,14], Bluman et al. considered the contact symmetries and higher-order symmetries of differential equations, such higher-order symmetries are generalized symmetries actually. In [15–18], the generalized conditional symmetries (GCS) are presented by the GCS ansatz method. However, the GCS ansatz represents some special types of generalized symmetries only, and it does not include the point symmetry. So we think that it is not complete in the sense of symmetry analysis. In [19–23], the symmetry-based integrability and integrable classification of the nonlinear evolution equations are investigated, but the symmetries of the equations are not given explicitly. Based on the generalized vector field theory, our method for dealing with generalized symmetries is systematic and complete. By this method, all of the generalized symmetries of the equations can be obtained in principle, and such generalized symmetry includes the point symmetry as its special case naturally.

We note that Eq. (1.1) contains several arbitrary parameters as its coefficients and exponents, such an equation differs from the equations with specific parameters, and the unknown parameters affect the symmetry and other property of the equation greatly, such as the integrable properties and exact solutions. So, the known results on the equations with specific parameters are not applicable for the equations with arbitrary parameters. Moreover, determination of the generalized symmetries of the equation with unknown parameters is a complicated problem that challenges researchers.

The main purpose of this paper is to develop the combination of generalized symmetry classification and the recursion operator method for dealing with integrable properties and exact solutions to Eqs. (1.1) and (1.2). The remainder of this paper is organized as follows: in Section 2, we perform generalized symmetry classification on Eq. (1.1), and present all of the second and third-order generalized symmetries of the equation. As its special case, the generalized symmetries of Eq. (1.2) are presented, and the fourth-order generalized symmetry of a nonlinear equation is considered. In the sense of generalized symmetry, the integrability of the general nonlinear equation is investigated by the symmetry classification method for the first time in the literature. In Section 3, the recursion operators of Eq. (1.2) are constructed, and the infinite number of generalized symmetries are provided accordingly. Then, the integrable properties of the equation are provided. In Section 4, the exact and explicit solutions to the equations generated from the generalized symmetries are investigated, most of them are new and interesting. Finally, the conclusions and some remarks are given in Section 5.

2. Generalized symmetry classifications for the nonlinear diffusion equations

In this section, we shall construct the generalized symmetries of the second-order nonlinear diffusion Eq. (1.1) in the evolution form

$$V = Q[u]\partial_u, \tag{2.1}$$

where $Q \equiv Q[u] = Q(x, t, u, u_x, \ldots)$ is the characteristic. Meanwhile, we shall develop the generalized symmetry method for dealing with the complete symmetry classifications of the general diffusion Eq. (1.1), then the complete generalized symmetry classifications of Eq. (1.2) are obtained successively.

If the generalized vector field (2.1) generates a symmetry of Eq. (1.1), then V must satisfy the following generalized symmetry condition:

$$\mathrm{pr}V[\Delta]|_{\Delta=0} = 0, \tag{2.2}$$

where $\Delta = u_t - \alpha u^p u_{xx} - \beta u^q u_x^2$, $\mathrm{pr}V$ denotes the infinite prolongation of V.

2.1. Generalized symmetries of the second-order characteristics

Firstly, we suppose that $Q[u] = Q(x, t, u, u_x, u_{xx})$ is the second-order characteristic of V. Then, by the method of characteristics, we have the following result:

Theorem 2.1. *For the second-order characteristic and arbitrary parameters α, β, p and q, if we assume that $\alpha\beta \neq 0$ and $p \neq 0$, then the complete symmetry classification of Eq. (1.1) is as follows:*