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Noether symmetries and conserved quantities for fractional Birkhoffian systems with time delay

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ABSTRACT

The Noether symmetries and the conserved quantities for fractional Birkhoffian systems with time delay in terms of Riemann–Liouville fractional derivatives are proposed and studied. First, the fractional Pfaff–Birkhoff principle with time delay is proposed, and the fractional Birkhoff's equations with time delay are obtained. Second, based on the invariance of the fractional Pfaff action with time delay under a group of infinitesimal transformations, the Noether symmetric transformations and the Noether quasi-symmetric transformations of the system are defined, and the criteria of the Noether symmetries are established. Finally, the relationship between the symmetries and the conserved quantities are studied, and the Noether theorems for fractional Birkhoffian systems with time delay are established. Some examples are given to illustrate the application of the results.

1. Introduction

The fractional calculus first appeared in the letter that L'Hôspital wrote to Leibniz in 1695. Shortly after being introduced, numerous new theories were proposed by many scholars [1–3]. In recent years, extensive applications about fractional calculus can be found in physics, mechanics, engineering, dynamics, control theory, modeling, probability, biology, chemistry, economics, etc. More significantly, the variational calculus is of great importance in some of those areas. Riewe [4,5] investigated a version of the Euler–Lagrange equations for the problems of calculus of variations with fractional derivatives. Riewe's work not only developed Lagrangian mechanics and Hamiltonian mechanics but also turned out to be very attractive to many mathematicians and physicists. Agrawal [6] studied the fractional variational problems in terms of Riemann–Liouville fractional derivatives and the corresponding Euler–Lagrange equations and boundary conditions were derived. Baleanu and Avkar generalized Agrawal's approach to study problems with Lagrangians which are linear on the velocities [7]. However, Atanacković et al. [8] investigated the Euler–Lagrange equations for a variational problem in which the lower bound of the functional is assumed to be different with the lower bound in the left Riemann–Liouville fractional derivative. What's more, many achievements have been made in research on fractional Lagrangian mechanics and fractional Hamiltonian mechanics [9–16]. Almeida and Malinowska [17], Klimek et al. [18] and Malinowska et al. [19] studied the advanced methods in the fractional calculus of variations systematically. The mathematical theories about the fractional calculus are still in development. Recently, some results about the theory of local fractional calculus which was considered to be one of the useful tools to deal with the fractal and continuously non-differentiable functions were obtained [20–23].

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The symmetry and the conserved quantity are important aspects of research to reveal dynamical characteristics of a dynamical system. Frederico and Torres proved the fractional Noether theorems [24] by introducing a new concept of fractional conserved quantity. However, Atanacković argued that the fractional conserved quantity that Frederico and Torres introduced is not constant in time, so the concept of conserved quantity is not clear. In view of this, Atanacković et al. [25] proposed another invariant condition of the fractional variational problems and established the fractional Noether theorems in terms of the concept of conserved quantity as it done in classical theory. Other works on Noether symmetries and conserved quantities for fractional variational problems are presented in [26–30].

However, in the state of nature, the phenomenon of time delay is almost universal, ranging from natural science, engineering technology to social science. A considerable amount of research has proved that even an existing time delay with millisecond on the behavior of a dynamical system produces complex dynamic responses [31–33]. In 1964, Èl'sgol'c [34] presented and studied the variational problems with time delay for the first time. Hughes [35] obtained the Euler–Lagrange equations by combining the optimal control problems with the variational problems with time delay. Furthermore, the mathematical aspects and the applications of the variational problems with time delay have been studied profoundly by numerous scholars [36–42]. The Noether symmetry theorems with time delay both in Lagrangian and Hamiltonian involved with the delayed optimal control systems were obtained by Frederico and Torres [43]. Recently, the Noether theorems for nonconservative dynamical systems with time delay [44], Hamiltonian systems with time delay [45] and Birkhoffian systems with time delay [46] were investigated respectively by Jin and Zhang and Zhai and Zhang. However, it's worth mentioning that the study of the Noether symmetries and the conserved quantities for fractional variational problems with time delay has just begun.

The fractional calculus model can be used to make a more accurate description and to give a deeper insight into the physical processes underlying long range memory behaviors. On the other hand, the time delay phenomenon incorporates a memory–time property because the evolution of a system not only depends on a certain time t but also the state of the system at an earlier time $t-\tau$, namely it seems that some kind of relationship between the fractional calculus and the dynamics with time delay exists, because both of them possess the properties of memory. Motivated by the above idea, Lima et al. [47] analyzed the relationship between the signal delay and the fractional dynamics from practical experiments. After that, Baleanu et al. investigated the fractional variational problems by combining time delay with Riemann–Liouville derivatives and Caputo derivatives [48,49], and corresponding Euler–Lagrange equations were obtained. The main advantage of using the combination of fractional derivatives and time delay is that for a fractional Lagrangian with time delay, non-locality introduced naturally both in fractional derivatives and the potential part. And this kind of fractional variational problems with time delay were further applied in optimal control problems [50,51].

This paper deals with the Noether symmetries and the conserved quantities for fractional Birkhoffian systems with time delay. As a more general dynamical system, the Birkhoffian mechanics [52–54] not only have a high generalization in theory but also it is applicable to classical mechanics and various types of constrained mechanical systems. What's more, it has more valuable and profound applications in a great many fields such as quantum mechanics, statistical mechanics, atomic and molecular physics and biological physics etc. [55,56]. And the study of Birkhoffian dynamics has made significant progress recently [57–62]. In Ref.[29], the invariance properties of Pfaff variational problems under the El-Nabulsi model were studied. And the preliminary work on fractional Birkhoff's equations and Pfaff–Birkhoff principle could be found in Ref.[60]. It is worth noting that the Birkhoffian systems with time delay have not been investigated yet in the form of fractional model in the literature.

This paper is organized as follows: In Section 2, we present some knowledge of fractional derivatives related to this paper. In Section 3, we establish the fractional Pfaff–Birkhoff principle with time delay and obtain the fractional Birkhoff's equations with time delay. Section 4 deals with the variation of the fractional Pfaff action with time delay. In Section 5, we establish the definitions and the criteria of the Noether symmetric transformations and quasi-symmetric transformations for the fractional Birkhoffian systems with time delay. In Section 6, we present the Noether theorems of fractional Birkhoffian systems with time delay and discuss the relationship between the Noether symmetries and the conserved quantities. In Section 7, two examples are given to illustrate the application of our main results. Finally, Section 8 is devoted to our conclusion.

2. Notation

In this section, let us present some definitions and properties of fractional derivatives [1– 3] related to this paper.

Let f and g are continuous functions in the interval $[t_1, t_2]$, and the left Riemann–Liouville fractional derivative is defined as

$$ {}_{t_1}D_t^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_{t_1}^{t} (t-\tau)^{n-\beta-1} f(\tau)\mathrm{d}\tau, \tag{1} $$

and the right Riemann–Liouville fractional derivative is

$$ {}_{t}D_{t_2}^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_{t}^{t_2} (\tau-t)^{n-\beta-1} f(\tau)\mathrm{d}\tau, \tag{2} $$

where $\Gamma(*)$ is the Euler gamma function, the order β fulfills $n-1 \leq \beta < n$ and n is a positive integer. Especially, if β becomes an integer, these derivatives become the usual derivatives

$$ {}_{t_1}D_t^{\beta} f(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\beta} f(t), \quad {}_{t}D_{t_2}^{\beta} f(t) = \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\beta} f(t), \quad (\beta = 1, 2, \ldots) \tag{3} $$