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Characterizing two-timescale nonlinear dynamics using finite-time Lyapunov exponents and subspaces

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Finite-time Lyapunov exponents and subspaces are used to define and diagnose boundary-layer type, two-timescale behavior in the tangent linear dynamics and to determine the associated manifold structure in the flow of a finite-dimensional nonlinear autonomous dynamical system. Two-timescale behavior is characterized by a slow-fast splitting of the tangent bundle for a state space region. The slow-fast splitting is defined using finite-time Lyapunov exponents and vectors, guided by the asymptotic theory of partially hyperbolic sets, with important modifications for the finite-time case; for example, finite-time Lyapunov analysis relies more heavily on the Lyapunov vectors due to their relatively fast convergence compared to that of the corresponding exponents. The splitting is used to characterize and locate points approximately on normally hyperbolic center manifolds via tangency conditions for the vector field. Determining manifolds from tangent bundle structure is more generally applicable than approaches, such as the singular perturbation method, that require special normal forms or other *a priori* knowledge. The use, features, and accuracy of the approach are illustrated via several detailed examples.

1. Introduction

The flow of a finite-dimensional autonomous nonlinear dynamical system with multiple timescales may have manifold structure. Characterizing this structure can facilitate simplified analysis and computation, and lead to greater understanding of the system behavior. The relevant timescales are most generally in the linear variational dynamics, i.e., tangent linear dynamics. Our objective is to diagnose two-timescale behavior in tangent linear dynamics with slow dynamics and both stable and unstable fast dynamics, and to compute the associated manifold structure in the flow of the nonlinear system. Because the intent is to analyze finite-time behavior, we first define two-timescale behavior in this context. Though we only directly consider two timescales and the associated normally hyperbolic center manifolds in this paper, the discussion and results are relevant to systems with more than two timescales and also to additional manifold structure, such as the center-stable and center-unstable manifolds relevant to the solution of certain boundary-value problems [4,22,47,55]. We do not consider systems with persistent fast oscillations.

Many of the methods available for computing invariant manifolds (i) operate off the linear structure at an equilibrium point or a periodic orbit [13], or (ii) require *a priori* knowledge of system coordinates adapted to the manifold structure, e.g. [51], or

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(iii) require *a priori* knowledge of a manifold that can be analytically or numerically continued to the manifold of interest, e.g. [7,49], or (iv) require that the manifold is transversally stable [17,39]. Our particular application context is flight guidance and control, and our motivation comes from the notable successes of the singular perturbation method [31,45] in providing insight and facilitating solution approximation with reduced-order models [44]. Geometric singular perturbation theory [15,28] clarifies the manifold structure in the flow associated with two-timescale behavior. The singular perturbation method is one means of obtaining the manifold structure, but it requires a special coordinate representation, i.e., normal form, with a small parameter, such that the manifold structure for the parameter value of interest can be obtained via matched asymptotic expansions. The singular perturbation method can be viewed as an analytical continuation method; however there is no general systematic method of obtaining the required normal form.

The situation of interest is when two-timescale behavior is suspected in a region of state space, perhaps based on simulation experience, and one wants a means of diagnosing whether or not there are two (or more) disparate timescales and, if there are, a means of characterizing the associated flow structure. In addition to requiring methodology that works away from equilibria and periodic orbits and does not require the singularly perturbed normal form, there is the challenge that, for many applications, the methodology must be effective when only finite-time behavior is considered. The approach addressed in this paper, which we refer to as finite-time Lyapunov analysis (FTLA), uses finite-time Lyapunov exponents (FTLEs) and the associated vectors (FTLVs), to diagnose two-timescale behavior and characterize the associated tangent bundle structure, and then uses invariance-based orthogonality conditions to locate and compute the associated manifold structure. Orthogonality conditions are used in the intrinsic low-dimensional manifold (ILDM) method [38] in the chemical kinetics context to compute a slow manifold, but the tangent bundle structure is determined by a means other than FTLA. Orthogonality conditions are also used for the computation of invariant manifolds in [49], but the tangent bundle structure is derived from a known neighboring manifold in a numerical continuation scheme.

FTLA is used in different ways in several application contexts. The body of work (see for examples [6,24,52,54]) on characterizing finite-time manifold structure in time-dependent velocity fields has connections with our work, though the target is a co-dimension one manifold that separates the flow and not two-timescale behavior. In particular the maximum FTLE field is used to determine Lagrangian coherent structures in fluid flows with time-dependent velocity fields [12,23,54,56] and to assess the stability of orbits in celestial mechanics [16,57]. FTLA is used to identify the fastest growing direction(s) of initialization errors in weather predictability theory [8,34,58,59].

FTLA is applied to systems with slow-fast behavior in [2,41,42,50]. In [41], Lyapunov analysis is proposed as a means of diagnosing timescales and suggesting adapted coordinates as an alternative to the singular perturbation approach. In [42], the ILDM and computational singular perturbation (CSP) [32,33] methods for slow-fast behavior are interpreted geometrically using Fenichel theory and the idea of using FTLE/Vs to improve the ILDM method is proposed. In [2] Lyapunov analysis is applied to periodic and chaotic attractors, as well as slow manifolds, and an approach for computing FTLVs is developed. Lorenz [36] seems to have been the first to use FTLA in analyzing a chaotic attractor. In [50], FTLA is used to identify the dimension of the attracting slow manifold along a trajectory. The application of FTLA to the solution of two-timescale boundary value problems related to optimal control is discussed in [4].

The main contributions of the present paper are to extend FTLA to the diagnosis and computation of normally hyperbolic center manifolds and to clarify more generally the definitions and procedures for meaningful finite-time tangent bundle splittings. Because the finite time is limited, it is crucial to define the tangent bundle splitting of interest in the fastest converging way and to clarify the finite time required to accurately approximate the invariant tangent bundle splitting. Guided by the theory of partially hyperbolic sets [25], a finite-time two-timescale set is defined, requiring spatial and temporal uniformity of the spectral gap between the slow and fast FTLEs. A *fast stable / slow / fast unstable* tangent bundle splitting is specified in terms of the FTLVs. The size of the spectral gap dictates the rate of exponential convergence of the tangent bundle splitting toward the desired invariant splitting, providing a guideline for how large the finite time needs to be. We account for both fast stable and fast unstable behavior and provide orthogonality conditions for approximately computing points on normally hyperbolic center[3] manifolds, whereas previous work on FTLA considered only attracting or repelling center manifolds. Several detailed examples are presented to illustrate and clarify the approach, and to demonstrate its feasibility and effectiveness in locating and approximating invariant center manifolds.

The paper is organized as follows. In Section 2, we specify the dynamical system to be considered and recall some definitions from geometry. Section 3 provides a perspective of the approach and supplements the introduction with background and perspective required to understand the goals and contributions of the present work in relation to other work. Section 4 covers Lyapunov analysis: first we briefly describe the asymptotic theory of partially hyperbolic sets; second we define finite-time Lyapunov exponents and vectors (FTLE/Vs) and describe their use for the identification of tangent space structure; third we address the convergence of the tangent space structure; and fourth we contrast the properties of the FTLE/Vs and their asymptotic counterparts. In Section 5 we define a finite-time two-timescale set and a finite-time center manifold. The procedure for applying the approach is given in Section 6. Section 7 contains detailed examples. Conclusions are given in Section 8.

[3] We alert the reader that our use of 'center' is in the broader sense of the partially hyperbolic theory [25] where center subspaces and manifolds are associated with smaller magnitude Lyapunov exponents that are not necessarily zero.