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Order preserving contact transformations and dynamical symmetries of scalar and coupled Riccati and Abel chains

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ABSTRACT

We identify contact transformations which linearize the given equations in the Riccati and Abel chains of nonlinear scalar and coupled ordinary differential equations to the same order. The identified contact transformations are not of Cole-Hopf type and are *new* to the literature. The linearization of Abel chain of equations is also demonstrated explicitly for the first time. The contact transformations can be utilized to derive dynamical symmetries of the associated nonlinear ODEs. The wider applicability of identifying this type of contact transformations and the method of deriving dynamical symmetries by using them is illustrated through two dimensional generalizations of the Riccati and Abel chains as well.

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1. Introduction

Solving a nonlinear ordinary differential equation (ODE) by transforming it to a linear ODE is one of the classical methods of finding solutions in the theory of differential equations. Such a study is called linearization/equivalence problem [1]. The linearization can be achieved by identifying a suitable transformation. The transformation may be a point transformation [2], a contact transformation [3], a Sundman transformation [4–6], a nonlocal transformation [7–9], or a generalized linearizing transformation [10], to name a few. The transformation can be identified by analyzing the structure of the ODE. As far as second order nonlinear ODEs are concerned, it has been shown that the general form of the nonlinear ODE which can be transformed to the free particle equation by an invertible point transformation should be at the most cubic in the first derivative and its coefficients have to satisfy two conditions involving their partial derivatives [11]. Subsequently it has been proved that any second order nonlinear ODE which admits maximal Lie point symmetries (eight) also satisfies the lineariziability criteria [11,12]. A systematic procedure is also available to construct invertible point transformations from the Lie point symmetries [13]. In other words, one can find a connection between invertible point transformations and Lie point symmetries. Linearization of a second order nonlinear ODE by Sundman transformation was studied by Durate et al. [4]. Unlike the case of invertible point transformation, in the present case, the new independent variable is considered in a nonlocal form [7]. The connection between Sundman symmetries and Sundman transformation was studied in detail by Euler et al. [6]. Recently, three of the present authors have introduced a generalized linearizing transformation in which the new independent variable is not only nonlocal in form but also involves derivatives [10]. However, the connection between generalized linearizing transformations and symmetries is

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not yet known. Apart from the above one can also find other linearizing transformations, for example, nonlocal transformations that can linearize a class of equations, see for example [14–17] and references therein. Again the interrelation between nonlocal transformations and symmetries is yet to be proved in general.

In this paper, we report a new kind of contact transformation which linearizes the given equation in the same order. From the literature it is known that the contact transformation linearizes a given ODE to a new ODE which is one order higher than the original one. The well known example is the modified Emden equation (MEE), $\ddot{x}+3x\dot{x}+x^3=0$. The contact transformation $x=\frac{\dot{w}}{w}$ linearizes the above equation to an equation of third order, that is $\dddot{w}=0$. On the other hand, as we show below, the above Liénard type nonlinear oscillator equation can also be linearized to the free particle equation $\frac{d^2u}{dt^2}=0$ through the contact transformation $u=\frac{x}{2\dot{x}+x^2}$. As one witnesses, the transformation does not increase the order of the underlying ODE. Unlike the known contact transformations (which are identified by an ad-hoc way) the ones which we report here can be derived in an algorithmic manner. An immediate consequence of finding this type of contact transformations is that one can look for dynamical symmetries associated with the underlying nonlinear ODEs. The latter result is significant since the dynamical symmetries are usually difficult to derive and no systematic method exists in the literature to explore them. This is because once the infinitesimal generators in the Lie symmetry analysis are allowed to involve derivative terms then one can no longer determine the complete symmetry group. Dynamical symmetries of ODEs can be regarded as the transformations of the set of first integrals [18,19]. Deviating from the conventional approach of solving the determining equations arising from the associated invariance condition, in this paper, we develop a simple and straightforward algorithm to derive dynamical symmetries associated with the given nonlinear ODE belonging to the Riccati and Abel chains of ODEs. Our work also reveals the hidden connection between contact transformations and dynamical symmetries. It is known that the MEE equation is the second member of the Riccati chain (see below), and the third member in this chain is a third order nonlinear ODE and the fourth member is a fourth order nonlinear ODE and so on [9]. The procedure which we develop here can also be extended to any of the equations in this chain. We also bring further light into this theory by considering the other chains, namely Abel chain and coupled Riccati and Abel chains. In all these chains the same methodology can be adopted to obtain the linearizing contact transformations and the dynamical symmetries of the given equation.

We note here that the linearization of the Riccati chain through the Cole-Hopf transformation is known. However, as far as the authors' knowledge goes, the linearization of any of the equations in the Abel chain has not been demonstrated so far. In this paper, we show the exact linearization of the equations admitted by the Abel chain through contact transformation. The integrable scalar Riccati and Abel chains are of the forms [9,20–24],

$$\left(\frac{d}{dt}+kx\right)^m x=0, \qquad m=1,2,3,\ldots \tag{1}$$

and

$$\left(\frac{d}{dt}+kx^2\right)^m x=0, \qquad m=1,2,3,\ldots \tag{2}$$

respectively. Interestingly one may also generalize the above differential equations to

$$\left(\frac{d}{dt}+f(x,t)\right)^m g(x,t)=0, \qquad m=1,2,3,\ldots. \tag{3}$$

The above two chains possess several interesting geometric structures and certain common mathematical properties, (see for example [20,25]). The results which we present here will further illuminate the shared mathematical properties of these two chains.

The methodology which we adopt to derive the aforementioned results is the following. In one of our earlier works [9] we have shown that the Riccati and Abel chains can be transformed to the linear equation

$$\frac{d^m u}{dt^m}=0, m=2,3,4\ldots \tag{4}$$

through the nonlocal transformation

$$u=g(t,x)e^{\int f(t,x)dt}. \tag{5}$$

For example, in the case of MEE we have $f=g=x$ and the transformed equation is just $\ddot{u}=0$. The general solution can be derived by solving the first order ODE [8], that is

$$\frac{\dot{u}}{u}=\frac{\dot{g}+gf}{g}, \tag{6}$$

which can be obtained from (5) itself. Since the left hand side of Eq. (6) is a function of t and is known from the linearized equation, we can rewrite the above equation as an equation for $\dot{x}$. Integrating the latter equation we can obtain the general solution of the nonlinear ODE. For more details one may refer to [7]. After analyzing the expression (6) carefully, we found that the contact transformation can also be captured from the identity (6) itself. To extract the contact transformation we split Eq. (6) into two separate expressions, one for u and the other for $\dot{u}$ with an unknown function in them (see Eq. (7a) given below).

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