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A numerical scheme and some theoretical aspects for the cylindrically and spherically symmetric sine-Gordon equations

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ABSTRACT

A finite difference formula based on the predictor–corrector technique is presented to integrate the cylindrically and spherically symmetric sine-Gordon equations numerically. Based on various numerical observations, one property of the waves of kink type is conjectured and used to explain their returning effect. Several numerical experiments are carried out and they are in excellent agreement with the existing results. In addition, the corresponding modulation solution for the two-dimensional ring-shaped kink is extended to that in three-dimension. Both numerical and theoretical aspects are utilized to verify the reliability of the proposed numerical scheme and thus the analytical modulation solutions.

1. Introduction

The nonlinear evolution equations play an important role in mathematical physics. It helps to describe the wave propagation process occurring in several physical contexts. The Korteweg-de Vries equation describes the uni-directed shallow water wave [20], while the Boussinesq equation allows the description of the bi-directed water wave [3,4]. The class of Klein–Gordon equations finds its application to classical and quantum mechanics [12,13,21,32]. Among them, the one-dimensional sine-Gordon equation can be found in the dynamics of crack according to the climbing of crystal dislocation in the Frenkel–Kontorova model [6,14]. Furthermore, a number of biological models can be associated with the sine-Gordon and the coupled sine-Gordon equations [18,19,37]. The multi-dimensional version of this equation also received much attention from physicists and mathematicians [1,2,7]. Specifically, the cylindrically and spherically symmetric sine-Gordon equations were numerically treated for the ring wave solutions and they exhibit several interesting behaviors. The returning effect for the cylindrically symmetric solitary wave was reported in the article [8], in which the author used the tag “rotationally”. Subsequently, the ring-shaped quasi-soliton and breather-like solutions of the sine-Gordon equation were examined and they are noticed to shrink after experiencing the returning effect [9,29]. The quasi-soliton is kink-shaped ring wave which propagates between the origin where it are reflected and a maximum radius where the returning effect occurs. Based on an heuristic

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Ansatz of the cylindrical or spherical kink-type solution, an analytical estimation of returning time was derived in the article [31]. As a consequence, it supplemented a reasonable verification of the constructed numerical solution. The shrinking phenomenon was then mathematically explained in association with the decay behavior of the solution [25]. An extensive investigation of the reflection behavior of the radially symmetrical kinks in the vicinity of the origin was conducted by Geicke [15–17]. According to these works, a mathematical explanation was given for the former numerical results. Further numerical experiments supported the hypothesis that the reflection effect leading to the reformulation of quasi-soliton with radiated energy occurs for the spherically but not circularly ring-shaped waves. Examining the solution around the origin during the reflection time, the notion of pulson was introduced. Pulson can be interpreted as the high-dimensional generalization of the soliton-antisoliton bound state solution "breather". Being different from the expanding and shrinking process of quasi-soliton, it corresponds to the oscillation of the solution in the vicinity of the origin [17,24]. Stability aspects of cylindrical pulsons were also intensively discussed but the corresponding mathematical reasoning was not yet given. Nevertheless, the fore-mentioned results were qualitatively presented and theoretically discussed in detail without providing any detailed numerical scheme. Much later, with advance of numerical techniques some new methods are proposed to retreat the two-dimensional sine-Gordon equation, though they can be generally applied to the three-dimensional version [5,11]. The gap between these works is still realizable. On the one hand, the former works reported the observation without providing the detail of numerical scheme in use and on the other hand, the latter did not link their numerical results to the qualitative behaviors. Our work builds a bridge connecting these works and also gives some insight into the quantitative knowledge on the returning effect from another perspective. Besides, based on the Whitham modulation theory for the multi-dimensional sine-Gordon equation developed in [23], the modulation solutions corresponding to the derivative of wave with respect to the radius coordinate are obtained. In addition to the analytical estimation of the returning time, these new results help to qualify the proposed numerical scheme. In this work we apply the predictor–corrector scheme to study the ring waves governed by the two-dimensional and three-dimensional sine-Gordon equations. These equations are solved in the setting of the polar and spherical coordinates instead of the Cartesian coordinates. From another point of view, the last part of this paper extends the results on the modulation theory from the previous works in [22,23].

2. Numerical method

2.1. Setting of the problem

The sine-Gordon equation is given by

$$u_{tt} - \Delta u + \sin u = 0, \quad (\Delta = \nabla \cdot \nabla). \tag{1}$$

The two-dimensional sine-Gordon equation in the polar coordinate (r, ϕ) can be written as

$$u_{tt} - \left(\frac{1}{r}u_r + u_{rr} + \frac{1}{r^2}u_{\phi\phi}\right) + \sin u = 0,$$

and the three-dimensional sine-Gordon equation in the spherical coordinate (r, θ, ϕ) reads

$$u_{tt} - \left(\frac{2}{r}u_r + u_{rr} + \frac{\sin\theta\, u_{\theta\theta} - \cos\theta\, u_\theta}{r^2 \sin\theta} + \frac{u_{\phi\phi}}{r^2\sin^2\theta}\right) + \sin u = 0.$$

For the cylindrically and spherically symmetric solutions that are independent of the angular coordinate, we consider

$$u_{tt} - \left(\frac{n_d - 1}{r}u_r + u_{rr}\right) + \sin u = 0,$$

where n_d is the number of dimension under consideration. It is clear that the difference between this equation in two- and three-dimensional space is only the constant n_d, so it is sufficient to develop a numerical scheme for the two-dimensional case and then to modify the appropriate coefficient to adapt to the one-dimensional and three-dimensional cases. For that reason, we shall exclusively derive the formulation in regard with the symmetric two-dimensional sine-Gordon equation with $n_d = 2$

$$u_{tt} - \left(\frac{1}{r}u_r + u_{rr}\right) + \sin u = 0. \tag{2}$$

We are going to solve this equation in a rectangular space-time domain $\Omega = [r_0, R] \times [t_0, T]$. It is assumed further that u is sufficiently differentiable during the course of derivation. Eq. (2) is complemented by the initial conditions of the given profile and velocity

$$u(r, t_0) = u_0(r), \quad u_t(r, t_0) = v_0(r) \quad \text{for } r_0 \le r \le R. \tag{3}$$

As we shall focus our attention on the ring wave of kink-type with the conservative energy over the whole domain, it is preferable to pose the conditions of no flux at two boundaries as follows

$$u_r(r_0, t) = u_r(R, t) = 0 \quad \text{for } t_0 \le t \le T. \tag{4}$$