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# An integral equation representation approach for valuing Russian options with a finite time horizon

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ABSTRACT

In this paper, we first describe a general solution for the inhomogeneous Black–Scholes partial differential equation with mixed boundary conditions using Mellin transform techniques. Since Russian options with a finite time horizon are usually formulated into the inhomogeneous free-boundary Black–Scholes partial differential equation with a mixed boundary condition, we apply our method to Russian options and derive an integral equation satisfied by Russian options with a finite time horizon. Furthermore, we present some numerical solutions and plots of the integral equation using recursive integration methods and demonstrate the computational accuracy and efficiency of our method compared to other competing approaches.



## 1. Introduction

A Russian option is a kind of path-dependent American option which entitles the holder to either buy or sell the underlying asset at the best price at which it is traded during the life of the option. Because American option holders can exercise their rights at any instant before maturity, the valuation of such options is usually classified as optimal stopping problems or free boundary problems. In addition, the value of these options contains an additional early exercise premium compared to European type options. A considerable amount of research has been conducted on options which combine features of American options with those of path-dependent options. For example, Dai and Kwok studied American floating strike lookback options [1], and Lai and Lim researched American fixed strike lookback options [2].

The Russian option was introduced by Shepp and Shiryaev in [3] and can be considered a type of perpetual American fixed strike lookback option. Ekström analyzed the regularity of the free boundary of Russian options with a finite time horizon and derived partial differential equations (PDEs) satisfied by these options [4]. Peskir drew out integral equations satisfied by Russian options with finite maturity using a stochastical local time-space formula [5]. An integral equation was first used to solve option pricing problems in the valuation of American options. Kim [6], was the first to derive the integral equations satisfied by the value of American options, which are known to have no closed form solutions. In general, it is not possible to solve such integral equations analytically; instead, numerical methods have to be found that would allow the solutions to be approximated. To date, there have been various numerical approaches. Huang et al. used recursive integration methods [7], and Ju [8] utilized the multipiece exponential function method to solve such integral equations numerically. The interested reader can refer to [9,10], which contain numerous approaches for solving American option

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problems from basic finite difference methods to methods using binomial trees. In this paper, we use the recursive iteration method proposed by Huang et al. [7] to obtain the numerical solution of the integral equation satisfied by Russian options, which we derive in subsequent sections.

A Russian option with a finite time horizon can be formulated into a parabolic PDE with mixed boundary conditions. Kimura [11], instead of solving the PDE directly, expressed the solution using a Laplace transform. In this paper, we derive integral equations satisfied by Russian options with a finite time horizon by solving the parabolic PDE directly using Mellin transform techniques. The Mellin transform is a type of integral transform and can be considered as a two-sided Laplace transform. Especially, it converts a Black–Scholes type PDE into a simple ordinary differential equation (ODE). Therefore, the use of the inverse Mellin transform enables the analytical representation of the value function of Russian options to be easily obtained. For this reason, the Mellin transform is widely used in option pricing. To list some examples, Panini first introduced option pricing using the Mellin transform [12,13], whereas Yoon and Kim obtained the closed solution for vulnerable options using double Mellin transforms [14]. Jeon et al. [15] drew a pricing formula of vulnerable geometric Asian options using time-dependent coefficients Black Scholes partial differential equation and Jeon et al. derived integral equations satisfied by American floating strike lookback options [16].

In this paper, we derive analytic solution for the inhomogeneous Black–Scholes equation with mixed boundary conditions by using Mellin transform approach. Mixed boundary condition usually arises in the option pricing problem when the underlying asset involves maximum process. We formulate Russian options as a PDE with mixed boundary conditions and obtain the integral equation satisfied by Russian option values by using the analytic formula we derived. We get numerical solutions for the integral equation by applying recursive iteration method, Also, we present the computational speed and accuracy of recursive integration by comparing its numerical results with some of existing methods in the literature.

This paper consists of five parts. In the first part (Section 2), we formulate the problem of valuing Russian options with a finite time horizon in terms of a free boundary PDE problem. In the second part (Section 3), we summarize the basic definition, properties, and lemma regarding the Mellin transform for those who are unfamiliar with it. In the third part (Section 4), we derive the general solution of the inhomogeneous Black–Scholes equation with mixed boundary conditions with the aid of Mellin transform techniques. In the fourth part (Section 5), we first perform a premium decomposition for Russian options, and obtain the integral equation satisfied by the free boundary for Russian options by applying the results of Section 4. In the fifth part (Section 6), we derive the closed form solution for the perpetual Russian options. In the last part (Section 7), we apply recursive integration methods to obtain a numerical solution for the integral equation we derived in Section 5 and analyze the solution qualitatively.

## 2. Model formulation: free boundary problem

The usual assumptions for the Black–Scholes option pricing framework are adopted in this work. Let $(S_t)_{t\geq 0}$ denote the price of an underlying asset of a Russian option under a risk-neutral probability measure $\mathbb{P}$. The stochastic dynamics of $S_t$ is described by

$$dS_t = (r-q)S_t dt + \sigma S_t dW_t, \quad S_0 = s \tag{2.1}$$

where $r > 0$ is the risk-free interest rate, $q \geq 0$ is the continuous dividend rate, and $\sigma > 0$ is the constant volatility of $S_t$. $W_t$ is a one-dimensional standard Brownian motion process on a filtered probability space $(\Omega, \mathcal{F}_{t\geq 0}, \mathbb{P})$, where $\mathcal{F}_{t\geq 0} \equiv \mathbb{F}$ is the natural filtration generated by $(W_t)_{t\geq 0}$. For the pricing process $(S_t)_{t\geq 0}$, we define the maximum process as

$$M_t = \Big(\max_{0\leq\gamma\leq t} S_\gamma\Big) \vee m \tag{2.2}$$

where $m \geq s > 0$ are given and fixed.

Consider a Russian option with a given finite time horizon $T > 0$. In the absence of arbitrage opportunities, the value $R(t, s, m)$ is a solution of the *optimal stopping problem*

$$R(t,s,m) = \sup_{0\leq\tau_t\leq T-t} \mathbb{E}\Big[e^{-r\tau_t} M_{\tau_t} \mid S_0 = s, M_0 = m\Big] \tag{2.3}$$

where $\tau_t$ is the stopping time of the filtration $\mathbb{F}$ and the conditional expectation is calculated under the risk-neutral probability measure $\mathbb{P}$. The random variable $\tau_t \in [0, T-t]$ is considered an optimal stopping time if it is able to provide the supremum value of the right hand side of (2.3).

Solving the optimal stopping problem (2.3) is equivalent to finding the points $(t, S_t, M_t)$ for which early exercise before maturity would be optimal.

Let

$$\mathcal{D} = \{(t,s,m) \in [0, T-t] \times (0, m] \times \mathbb{R}_+\} \tag{2.4}$$

Then, domain $\mathcal{D}$ of the pricing model can be divided into two regions: the stopping region $\mathcal{S} = \{(t,s,m) \in \mathcal{D} \mid 0 < s < s^*(t,m)\}$, and the continuation region $\mathcal{S}^C = \{(t,s,m) \in \mathcal{D} \mid s^*(t,m) < s \leq m\}$. Here, $s^*(t, m)$ is termed the *free boundary* or *early exercise boundary* and is given by

$$s^*(t,m) = \inf\Big\{s \in [0,m] \mid (t,s,m) \in \mathcal{S}^C\Big\}$$