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Heterogeneous, weakly coupled map lattices

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ABSTRACT

Coupled map lattices (CMLs) are often used to study emergent phenomena in nature. It is typically assumed (unrealistically) that each component is described by the same map, and it is important to relax this assumption. In this paper, we characterize periodic orbits and the laminar regime of type-I intermittency in *heterogeneous* weakly coupled map lattices (HWCMLs). We show that the period of a cycle in an HWCML is preserved for arbitrarily small coupling strengths even when an associated uncoupled oscillator would experience a period-doubling cascade. Our results characterize periodic orbits both near and far from saddle–node bifurcations, and we thereby provide a key step for examining the bifurcation structure of heterogeneous CMLs.

1. Introduction

Numerous phenomena in nature — such as human waves in stadiums [1] and flocks of seagulls [2] — result from the interaction of many individual elements, and they can exhibit fascinating emergent dynamics that cannot arise in individuals or even in systems with a small number of components [3]. In practice, however, a key assumption in most such studies is that each component is described by the same dynamical system. However, systems with heterogeneous elements are much more common than homogeneous systems. For example, a set of interacting cars on a highway that treats all cars as the same ignores different types of cars (e.g., their manufacturer, their age, different levels of intoxication among the drivers, etc.), and a dynamical system that governs the behavior of different cars could include different parameter values or even different functional forms entirely for different cars. Additionally, one needs to use different functional forms to address phenomena such as interactions among cars, traffic lights, and police officers. Unfortunately, because little is known about heterogeneous interacting systems [4,5], the assumption of homogeneity is an important simplification that allows scholars to apply a plethora of analytical tools. Nevertheless, it is important to depart from the usual assumption of homogeneity and examine coupled dynamical systems with heterogeneous components.

The study of coupled map lattices (CMLs) [6,7] is one important way to study the emergent phenomena (e.g., cooperation, synchronization, and more) that can occur in interacting systems. CMLs have been used to model systems in numerous fields, ranging from physics and chemistry to sociology, economics, and computer science [7–11]. In a CML, each component is a discrete dynamical system (i.e., a map). There are a wealth of both theoretical and computational studies of homogeneous CMLs [6,7,12–18], in which the interacting elements are each governed by the same map. Such investigations have yielded insights on a wide variety of phenomena. As we mentioned above, the assumption of homogeneity is a major simplification that often is not justifiable. Therefore, we focus on heterogeneous CMLs, in which the interacting elements are

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governed by different maps or by the same map with different parameter values. The temporal evolution of a heterogeneous coupled map lattice (CML) with p components is given by

$$X_i(n+1) = f_{R_i}(X_i(n)) + \varepsilon \sum_{\substack{h=1 \\ h\neq i}}^{p} f_{R_h}(X_h(n)), \quad i \in \{1,\dots,p\}, \tag{1}$$

where $X_i(n)$ represents the state of the system at instant n at position i of a lattice and $\varepsilon > 0$ weights the coupling between the different entities in the system. We consider entities in the form of oscillators, where the ith oscillator evolves according to the map

$$X_i(n+1) = f_{R_i}(X_i(n)), \qquad i \in \{1,\dots,p\}, \tag{2}$$

where the f_{R_i} are, in general, different functions that depend on a parameter R_i (where $i \in \{1,\dots,p\}$). We assume that each f_{R_i} is a C^2 unimodal function that depends continuously on the parameter R_i and that there is a critical point C at R_i. As usual, f^m means that f is composed with itself m times. If an uncoupled oscillator $X_i(n)$ takes the value $x_{i,n}$, then the evolution of this value under the map is $x_{i,n+1} = f_{R_i}(x_{i,n})$.

In this paper, we examine heterogeneous, weakly coupled map lattices (HWCMLs). Weakly coupled systems can exhibit phenomena (e.g., phase separation because of additive noise [19]) that do not arise in strongly coupled systems, and one can even use weak coupling along with noise to fully synchronize nonidentical oscillators [20]. Thus, it is important to examine HWCMLs, which are amenable to perturbative approaches. In our paper, we characterize periodic orbits both far away from and near saddle–node (SN) bifurcations. Understanding periodic orbits is interesting by itself and is also crucial for achieving an understanding of more complicated dynamics (such as chaos) [21,22]. We also characterize the laminar regime of type-I intermittency in our HWCMLs. Finally, we summarize our results and briefly comment on applications.

2. Theoretical results

Before discussing our results, we need to define some notation. Let $x_{i,n|R_i}$ denote the points in a periodic orbit of the ith uncoupled oscillator with control parameter R_i. The parameter value r_i is a bifurcation value of R_i for the ith map, so $x_{i,n|r_i}$ denotes the points in a periodic orbit at this parameter value.

Suppose that $R_i = r_i + \varepsilon^{\alpha}$, where ε is the same as in the coupling term of the CML (1) and $\alpha \in (0, \infty)$ is a constant. We seek to derive results that are valid at size $O(\varepsilon)$. We need to consider the following situations:

- $\alpha < 1$: In this case, when we expand to size $O(\varepsilon)$, the coupling term does not contribute at all. Therefore, the oscillators in (1) behave as if they were uncoupled at this order of the expansion.
- $\alpha > 1$: In this case, the coupling term controls the ε bifurcation terms. Thus, to size $O(\varepsilon)$, we cannot study the behavior of the bifurcation.
- $\alpha = 1$: In this case, we are considering a perturbation of the same size as the coupling term, and we can simultaneously study the coupling and the bifurcation analytically.

To study orbits close to bifurcation points, we thus let $R_i = r_i + \varepsilon$, where ε is the same as in the coupling term of the CML (1). In our numerical simulations (see Section 3), we will also briefly indicate the effects of considering $\alpha \neq 1$ (see Section 3.3).

2.1. Study of the CML far from and close to saddle–node bifurcations

In this section, we examine heterogeneous CMLs in which the uncoupled oscillators have periodic orbits either far from or near SN bifurcations. As periodic orbits exhibit different dynamics from each other depending on whether they are near or far from SN bifurcations [23,24], it is important to distinguish between these two situations.

A period-m SN orbit is a periodic orbit that is composed of m “SN points” of the composite map $f^m_{r_i}$. Each of these m SN points is a fixed point of $f^m_{r_i}$ at which $f^m_{r_i}$ undergoes an SN bifurcation. Period-m SN orbits play an important role in a map's bifurcation structure, because they occur at the beginning of periodic windows in bifurcation diagrams. Studying them is thus an important step towards examining the general bifurcation structure of a map.

When $f^m_{r_i}$ undergoes an SN bifurcation, the map f_{r_i} has two properties that we highlight. Let $\{x_{i,1|r_i}, x_{i,2|r_i}, \dots, x_{i,m|r_i}\}$ be a period-m SN orbit. It then follows that:

1. We have the relation

$$\frac{\partial f^m_{r_i}}{\partial x}(x_{i,j|r_i}) = 1 = \prod_{k=j}^{j+m-1} \frac{\partial f_{r_i}}{\partial x}(x_{i,k|r_i}).$$

Consequently, orbits that are near an SN orbit satisfy

$$1 - \prod_{k=j}^{j+m-1} \frac{\partial f_{R_i}}{\partial x}(x_{i,k|R_i}) = o(1). \tag{3}$$