Commun Nonlinear Sci Numer Simulat 36 (2016) 58–64

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Short communication

# Entropy analysis of systems exhibiting negative probabilities

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ARTICLE INFO

*Article history:*
Received 27 September 2015
Revised 20 November 2015
Accepted 23 November 2015
Available online 1 December 2015

*Keywords:*
Entropy
Negative probability
Fractional calculus
Quasiprobability distributions

ABSTRACT

This paper addresses the concept of negative probability and its impact upon entropy. An analogy between the probability generating functions, in the scope of quasiprobability distributions, and the Grünwald–Letnikov definition of fractional derivatives, is explored. Two distinct cases producing negative probabilities are formulated and their distinct meaning clarified. Numerical calculations using the Shannon entropy characterize further the characteristics of the two limit cases.



## 1. Introduction

Kolmogorov formalized present day probability theory where values go from 0 to 1 [1]. The concept of Negative Probability (NP) seems 'inadequate' for real world application, but the notion emerged as a valuable tool in the scope of quantum physics.

Dirac [2] introduced the NP when addressing the idea of negative energies. He noted that *'Negative energies and probabilities should not be considered as nonsense. They are well-defined concepts mathematically, like a negative of money'*. Feynman [3,4] also explored the NP and probabilities with value above 1. Feynman noted that no one objects the use of negative numbers in calculations, but that *'minus three apples'* is not valid in the real world. NP has been applied in physics [5–18], however the topic remained overlooked in other scientific fields.

In the area of mathematics we can mention the efforts of Bartlett [19] towards a formal definition of NP. Székely [20] introduced the *'half-coins'* as conceptual objects leading to NP. The topic became more popular during the last years and we verify the publishing of new works [21–23].

In the area of applied sciences we can mention Haug [24], that used NP in mathematical finance, Tijms and Staats [25], that adopted NP in waiting-time probabilities, Burgin and Meissner [26], with NP in financial option pricing, and Machado [27] in control.

Entropy plays a relevant role in thermodynamics and information theory [28–35]. This important statistical index evolved also towards considering probabilities outside the standard range between 0 and 1 [36–41], yet the evaluation of the impact of NP upon entropy is still far from being understood.

In this paper the concept of NP is addressed and an analogy with fractional derivatives [42–44] is established. This formulation leads to a deeper insight towards the underlying fundamental concepts involved with NP [27,45]. Furthermore, based on the synergies of the two fields, the paper explores the application of entropy with quasiprobability distributions [46,47].

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http://dx.doi.org/10.1016/j.cnsns.2015.11.022

Bearing these facts in mind, the paper is organized as follows. Section 2 introduces the fundamental aspects involving NP and, correspondingly, the generalization of entropy. Section 3 presents the results for several numerical experiments based on the new ideas. Finally, Section 4 outlines the main conclusions.

## 2. Fundamental concepts

For a discrete random variable $X$, with sample space $\Omega$, the probability generating function (pgf) is defined as:

$$G_X(z) = E\left(z^X\right) = \sum_{\nu=0}^{\infty} P(X = x_\nu) z^\nu, \ z \in \mathbb{C}, \tag{1}$$

where $P(X = x_\nu)$ is the probability mass function of $X$.

An important property is that the addition of independent random variables $Z = X + Y$ corresponds to the multiplication of their pgf's $G_Z(z) = G_X(z) \cdot G_Y(z)$. In particular, $Z = \sum_{\nu=1}^{n} X$, $n \in \mathbb{N}$, has pgf $G_Z(z) = [G_X(z)]^n$.

Székely [20] used the pgf a path toward NP, supported by the mind experiment of one coin toss. For a fair coin we have the sample space $\Omega = \{0, 1\}$ and the uniform probability mass function. Therefore, the pgf for $n$ coins tosses is simply $G_X(z) = \left[\frac{1}{2}(1+z)\right]^n$. Székely proposed for *'half-coin'* the event producing the pgf:

$$G_X(z) = \left[\frac{1}{2}(1+z)\right]^{\frac{1}{2}}. \tag{2}$$

If we flip the strange object twice, then the sum of the outcomes is either 0 or 1, with probability $\frac{1}{2}$, as if we had flipped once a standard coin.

Expanding expression (2) we obtain:

$$G_X(z) = \frac{1}{\sqrt{2}}\left(1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{1}{16}z^3 - \frac{5}{128}z^4 + \cdots\right). \tag{3}$$

The exotic object reveals an infinite number of faces, some having NP since $P(X = k) = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}$, $k = 0, 1, 2, \ldots,$ can take positive and negative values for $n = \frac{1}{2}$.

NP occur in the scope of quasiprobability distributions [46,47]. Quasiprobability distributions share many of the features of standard probabilities (e.g., the expectation value), but they violate the first and third of Kolmogorov's axioms of probability theory, since NP may occur and regions integrated under them do not represent probabilities of mutually exclusive states.

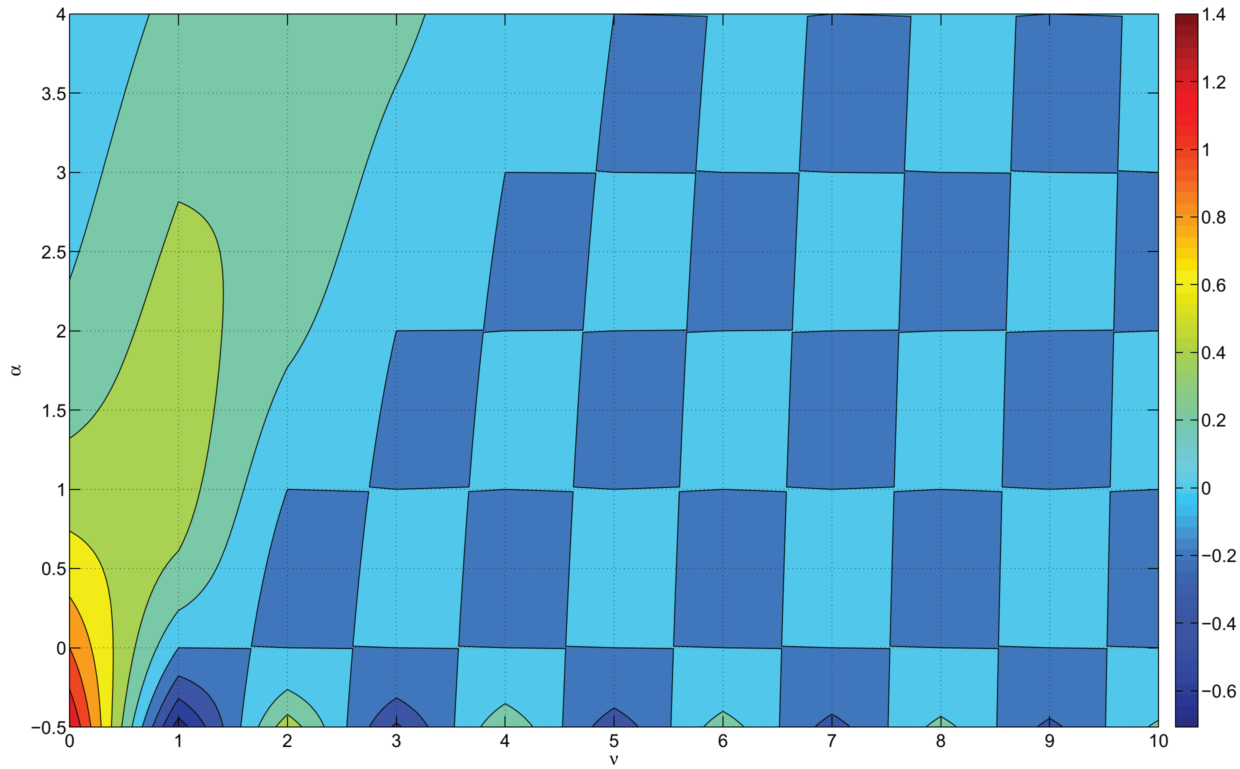


**Fig. 1.** Quasiprobability distribution for case $\mathcal{A}$, *'fractional toss of the coin'*, versus $(\nu, \alpha)$, for $-0.5 \le \alpha \le 4$.