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Direct functional separation of variables and new exact solutions to axisymmetric unsteady boundary-layer equations

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ABSTRACT

The paper deals with equations describing the unsteady axisymmetric laminar boundary layer on an extensive body of revolution as well as axisymmetric jet flows. Such equations are shown to reduce to a single nonlinear third-order PDE with variable coefficients

$$w_{tz} + w_z w_{xz} - w_x w_{zz} = \nu z w_{zzz} + F(t, x),$$

where w is a modified stream function. We describe a number of new generalized and functional separable solutions to this equation, which depend on two to four arbitrary functions of a single argument (a few solutions depend on an arbitrary function of two arguments). We use three methods to construct the exact solutions: (i) direct method for symmetry reductions, (ii) direct method of functional separation of variables (a special form of solutions with six undetermined functions is preset and particular solutions to an auxiliary ODE are used), and (iii) method of generalized separation of variables. Most of the solutions obtained are expressed in terms of elementary functions, provided that the arbitrary functions are also elementary. Such solutions, having relatively simple form and presenting significant arbitrariness, can be especially useful for testing numerical and approximate analytical methods for nonlinear hydrodynamic-type PDEs and solving certain model problems. The direct method of functional separation of variables used in this paper can also be effective for constructing exact solutions to other nonlinear PDEs.

1. Introduction. The classes of equations considered

1.1. Preliminary remarks

Hydrodynamic boundary-layer equations are important and fairly common in various areas of science and engineering (e.g., see [1–4]).

Exact solutions to the Navier–Stokes, boundary-layer, and related equations contribute to better understanding of qualitative features of steady and unsteady fluid flows at large Reynolds numbers; these features include stability, non-uniqueness, spatial localization, blow-up regimes, and others.

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Exact solutions with significant functional arbitrariness are of particular interest because they may be used as test problems to ensure efficient estimates of the domain of applicability and accuracy of numeric, asymptotic, and approximate analytical methods for solving suitable nonlinear hydrodynamic-type PDEs as well as certain model problems.

To find exact solutions to the Navier–Stokes, boundary-layer, and related equations, one usually employs the classical method for symmetry reductions [5–11] (based on the Lie group analysis of PDEs), direct method for symmetry reductions [12–17] (also known as the Clarkson–Kruskal direct method), nonclassical method for symmetry reductions [18–20], and method of generalized separation of variables [21–28]. For some other, less common methods, see [29–35]. Extensive surveys of exact solutions to the Navier–Stokes and boundary-layer equations can be found in [26,36–38].

The present paper looks for exact solutions to unsteady boundary-layer equations and employs the following three methods: (i) direct method for symmetry reductions, (ii) direct method of functional separation of variables, based on presetting the form of solutions and using particular solutions to an auxiliary ODE (this recent method was shown to be highly effective in [39]), and (iii) method of generalized separation of variables.

Remark 1. For methods allowing the construction of functional separable solutions to nonlinear PDEs, see, for example, [40–45]; see also [26,46] and references in them.

1.2. Plane boundary-layer equations

The system of unsteady plane laminar boundary-layer equations is written as [1,2]:

$$U_t + UU_x + VU_y = \nu U_{yy} + F(t, x), \tag{1}$$

$$U_x + V_y = 0 \tag{2}$$

where t is time, x and y are longitudinal (streamwise) and transverse coordinates (tangential and normal to the body surface), U and V are the longitudinal and transverse fluid velocity components, $F(t, x) = -p_x/\rho$ is a given function, p is the pressure, ρ is the mass density, and ν is the kinematic viscosity of the fluid. The fluid is assumed to be incompressible.

With the stream function W defined by

$$U = W_y, \quad V = -W_x, \tag{3}$$

system (1), (2) reduces to a single nonlinear third-order equation [1,2]:

$$W_{ty} + W_y W_{xy} - W_x W_{yy} = \nu W_{yyy} + F(t, x). \tag{4}$$

Exact solutions and transformations of Eq. (4) as well as different problems on the hydrodynamic boundary layer have been addressed by many researchers. For exact solutions to this equation in the steady case (with $W_t = 0$), see the papers [1–3,6,13,15,16,20,21,26]. Some exact solutions and transformations of the unsteady plane boundary-layer Eq. (4) can be found in [7,8,10,12,14,22,23,26,32–34].

1.3. Axisymmetric unsteady laminar boundary-layer equations on an extensive body of revolution

The system of axisymmetric unsteady laminar boundary-layer equations has the form [1,47]

$$U_t + UU_x + VU_r = \nu\big(U_{rr} + r^{-1}U_r\big) + F(t, x), \tag{5}$$

$$U_x + V_r + r^{-1}V = 0, \tag{6}$$

where U and V are the axial and radial components of the fluid velocity, respectively, while x and r are the axial and radial coordinates, with the other notations remaining the same as in Eq. (4). System (5), (6) describes an axisymmetric jet flow ($F \equiv 0$) or a boundary layer on an extensive body of revolution ($F \not\equiv 0$).

A self-similar solution to the steady Eqs. (5) and (6) with $F(t, x) \equiv 0$ for a jet flow problem was obtained in [1].

By changing to the new variables

$$U = 2r^{-1}w_r, \quad V = -2r^{-1}(w_x - \nu), \quad z = \frac{1}{4}r^2, \tag{7}$$

where w is a modified stream function, one reduces system (5), (6) to a single nonlinear third-order equation

$$w_{tz} + w_z w_{xz} - w_x w_{zz} = \nu z w_{zzz} + F(t, x). \tag{8}$$

Note that transformation (7) is slightly different from that used in [16,20,26] and leads to a simpler Eq. (8). The studies [16,20,26] present a number of transformations as well as exact solutions to the equation obtained from (8) by the change of variable $w = \bar{w} + \nu x$.

Remark 2. The studies [17,28,39] present one- and two-dimensional reductions and exact solutions to equations describing the unsteady axisymmetric boundary layer on an initial part of a body of revolution. The stream function equation dealt with in [39] can formally be obtained from Eq. (8) by replacing zw_{zzz} with $r^2(x)w_{zzz}$, where $r(x)$ is a dimensionless function that defines the shape of the body.

In subsequent sections, we will construct new exact solutions to the unsteady nonlinear third-order PDE (8), in which allowable forms of the pressure gradient function $F(t, x)$ will, as usual, be determined during the analysis.