Commun Nonlinear Sci Numer Simulat 31 (2016) 108–123

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Stability regions for fractional differential systems with a time delay

Jan Čermák*, Jan Horníček, Tomáš Kisela

Institute of Mathematics, Brno University of Technology, Technická 2, Brno CZ-616 69, Czech Republic

ARTICLE INFO

Article history:
Received 4 March 2015
Revised 21 July 2015
Accepted 22 July 2015
Available online 29 July 2015

Keywords:
Fractional delay differential equation
Stability
Asymptotic behaviour

ABSTRACT

The paper investigates stability and asymptotic properties of autonomous fractional differential systems with a time delay. As the main result, necessary and sufficient stability conditions are formulated via eigenvalues of the system matrix and their location in a specific area of the complex plane. These conditions represent a direct extension of Matignon's stability criterion for fractional differential systems with respect to the inclusion of a delay. For planar systems, our stability conditions can be expressed quite explicitly in terms of entry parameters. Applicability of these results is illustrated via stability investigations of the fractional delay Duffing's equation.

1. Introduction

Many problems described via ordinary differential equations require the inclusion of time delay terms. There are various types of these delays (technological, transport, incubation and others) appearing in many science areas. Delays are often involved in chemical processes (behaviours in chemical kinetics), technical processes (electric, pneumatic and hydraulic networks), biosciences (heredity in population dynamics), economics (dynamics of business cycles) and other branches. In general, the distinguishing feature of corresponding mathematical models is that the evolution rate of these processes depends on the past history. The differential equations modelling these problems are called delay differential equations (DDEs). Basic qualitative theory of these equations is well-established, especially in the linear case (for general references see [1] and [2]).

In the last decades, mathematical descriptions via fractional differential equations (FDEs) involving derivatives of non-integer orders turned out to be a very useful tool in the modelling of various phenomena of viscoelasticity, anomalous diffusion, control theory and other areas. Because of more degrees of freedom, fractional-order models are highly successful in situations where non-Gaussian and non-Markovian processes occur. A survey of interesting applications as well as basic qualitative properties of FDEs can be found, e.g. in the monographs [3–5].

Both these types of differential equations have a similar historical background. Although basic notions and properties related to these equations were discussed already by the founders of classical calculus, a systematic study of these equations was missing. It was only a rich application potential which gave rise to an enormous interest of mathematicians, engineers and other scientists in the qualitative and numerical investigations of DDEs and FDEs (starting in sixties of the last century).

The unification of DDEs and FDEs is provided by fractional delay differential equations (FDDEs), involving both the delay and non-integer derivative terms and disposing of great complexity. In technical applications, this approach is useful in creating strongly realistic models of certain processes and systems with memory and heredity. It is involved, among others, in analysis

* Corresponding author. Tel.: +420 5411 42535.
E-mail addresses: cermak.j@fme.vutbr.cz (J. Čermák), y115594@stud.fme.vutbr.cz (J. Horníček), kisela@fme.vutbr.cz (T. Kisela).

http://dx.doi.org/10.1016/j.cnsns.2015.07.008
1007-5704/

of various time delay systems whose stabilisation and control is realised via a state feedback. Following the recent trend, these controllers are often proposed using the fractional-order integro-differentiation method which gives rise to various types of FDDEs (e.g., for a fractional delay state space model of PD^{α} control of Newcastle robot we refer to [6]). This originally purely theoretical idea of fractional-order controllers was recently supported and justified by several papers studying experimentally collected impedance data of supercapacitors whose underlying electrochemical dynamics can be described and captured using of fractional-order models (see, e.g. [7] and the papers cited therein).

The stability issue is standardly of the main interest in the control theory and other areas where DDEs and FDEs play a significant role. The presence of time delay terms in feedback control systems results in characteristic equations involving transcendental terms of the exponential type. The corresponding stability polynomials (usually called quasi-polynomials) have infinitely many isolated zeros and analysis of their location (necessary for stability analysis of time invariant delay systems) is often a complicated matter. If these systems involve non-integer derivative terms, then the situation becomes even more difficult. As noted by several authors (see, e.g. [8]), the existing stability conditions for FDDEs do not provide effective algebraic criteria or algorithms for testing of stability of FDDEs and they are difficult to use in practice (see, e.g. [9]). From this viewpoint, the lack of such algebraic algorithms has hindered the advance of FDDEs for designs of control systems. Therefore, the main goal of this paper is to fill in this gap and present a simple and easily verifiable criterion for stability testing of linear time invariant fractional delay systems, including its rigorous mathematical justification (which seems to be extendable also to more general types of FDDEs).

The paper is organised as follows. In Section 2, we formulate our main result, namely effective stability criterion for a linear system of FDDEs. A brief survey of existing results on this topic is included as well. Section 3 shortly recapitulates a necessary background, mainly concerning fractional calculus and the Laplace transform. Section 4 discusses some properties of the studied system of FDDEs which are important in our investigations. In Section 5, we consider the characteristic equation for this system and analyse locations of its zeros in the complex plane. Using previous auxiliary assertions, Section 6 completes the proof of our main stability criterion. In Section 7, we reformulate this criterion for the corresponding planar system in terms of trace and determinant of the system matrix. Also, we apply the obtained results to stability investigations of the fractional delay Duffing's model of a nonlinear oscillator. Some remarks concerning future perspectives conclude the paper.

2. The main result

In the linear case, the fractional delay system

$$\mathrm{D}_0^{\alpha} y(t) = Ay(t-\tau), \quad t \in (0, \infty), \tag{2.1}$$

where D_0^{α} is the Caputo derivative of a real order $0 < \alpha < 1$, $A \in \mathbb{R}^{d \times d}$ is a constant real $d \times d$ matrix, $y : (-\tau, \infty) \to \mathbb{R}^d$ and $\tau > 0$ is a constant real lag, may serve as the basic prototype of FDDEs. The standard initial condition associated with (2.1) is

$$y(t) = \phi(t), \quad t \in [-\tau, 0] \tag{2.2}$$

where $\phi(t) \in L^1([-\tau, 0])$, i.e. all components of $\phi(t)$ are absolutely Riemann integrable on $[-\tau, 0]$.

Contrary to DDEs and FDEs, a general qualitative theory of FDDEs is just at the beginning. During the past years, several pioneering works on this topic appeared, with an emphasis put on stability issues. We recall that the zero solution of (2.1) is said to be stable (asymptotically stable) if for any $\phi(t) \in L^1([-\tau, 0])$ the solution $y(t)$ of (2.1), (2.2) is bounded (tends to zero as $t \to \infty$). Sufficient asymptotic stability conditions for (2.1) with multiple fractional orders and multiple delays have been formulated in [9] and [10] via the zeros location of associated characteristic equations. In the scalar case, asymptotic stability properties of

$$\mathrm{D}_0^{\alpha} y(t) = \lambda y(t-\tau), \quad t \in (0, \infty), \tag{2.3}$$

where λ is a real number, have been discussed in [11] by use of a transcendent inequality involving the fractional Lambert function. The bounded input–bounded output (BIBO) stability regions for (2.3), involving also the non-delayed term $y(t)$ on its right-hand side, have been investigated in [12] where the D-decomposition method was employed to describe the stability boundary of the studied equation in the form of parametric equations.

The first explicit stability criterion for (2.3) appeared in Theorem 5.1 of [13]. We recall here the relevant result in the following assertion (the symbol $\sim$ used here means an asymptotic equivalence in the sense that the ratio of both involved terms tends to a nonzero finite constant).

Theorem 1. *Let $\lambda \in \mathbb{R}$, $0 < \alpha < 1$ and $\tau > 0$.*
(i) The zero solution of (2.3) is asymptotically stable if and only if

$$-\left(\frac{\pi - \alpha\pi/2}{\tau}\right)^{\alpha} < \lambda < 0. \tag{2.4}$$

Moreover, $y(t) \sim t^{-\alpha}$ as $t \to \infty$ for any solution $y(t)$ of (2.3).
(ii) The zero solution of (2.3) is stable if and only if

$$-\left(\frac{\pi - \alpha\pi/2}{\tau}\right)^{\alpha} \le \lambda \le 0. \tag{2.5}$$

Note that for $\alpha = 1$, the condition (2.4) becomes $-\pi/2 < \lambda\tau < 0$ which is the classical asymptotic stability criterion for the first order delay equation $y'(t) = \lambda y(t-\tau)$.