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Modeling uncertainty in limit order execution

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ABSTRACT

A model for limit order execution is developed where several sources of uncertainty are taken into account. We focus on the optimal trading strategy of an investor who has to buy a block of shares throughout the submission of limit orders. This trading problem is explicitly solved and we analyze how the state of the limit order book and the investor's subjective beliefs affect the optimal strategy.

1. Introduction

The issue of how agents make their decisions in electronic financial markets is at the center of the recent research in market microstructure and modern investment theory. Nowadays most exchange systems are organized as order-driven electronic markets handling the whole trading order processing, from the submission of trade orders to their settlement. Trade orders are collected in limit order books (LOBs), where they are ranked according to some rules relating to the price and time priority of each order. The state of a limit order book crucially affects an incoming trader's decision regarding the type of order to submit, the aggressiveness and speed of the order submission process, how to adjust subsequent order placement depending on the successful implementation of previous orders, how to minimize the implementation cost, etc. All this poses new challenges in terms of modeling the sources of risk of the trading environment, and of solving the optimization problem which is faced by an agent. Here we focus on the problem of a buyer who has a block of shares, $\bar{X}$, to purchase within a time horizon, T, and submits some limit orders throughout an LOB. Limit orders are buy/sell orders that are proposed at a specified limit price—or at a better one—and are executed against a market order when the limit price is reached. The specification of a limit order includes the buy/sell attribute, the proposed limit price and the order size. The aggressiveness of an order is straightforwardly related to its limit price: a buy order above the best bid is more aggressive than one below the best bid. Aggressive limit orders are submitted because they increase the probability of getting filled[1], while favorable limit prices are associated with a smaller execution probability. With this type of orders, the main risk is to incur the lost opportunity costs of unfilled orders and adverse price movements while the orders await execution. Therefore, a main ingredient in a problem of optimal trading throughout limit orders is a model for the probability of order execution. In this paper we adopt an exponential function which is decreasing in the distance between the prevailing quote, S, and the limit price, $\hat{S}$, that is, the probability of execution is proportional to $\exp(-\lambda(S-\hat{S}))$, where the parameter $\lambda \geq 0$ represents the easiness of order execution. Such a parameter can in turn be related to LOB's characteristics, for example, the thicker the book, the larger the λ. While this functional shape is supported both in theoretical (Huitema [11],

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[1] Bias et al. [4] and Ranaldo (2006) [15] find evidence that buyers more frequently submit limit orders within the prevailing quotes, i.e. aggressive limit orders.

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Guéant et al. [9]) and empirical literature (Lo et al. [13]), in practice, an exact estimate of this parameter is not an easy task. Thus, following an intuition in Rickard and Torre [16], in the last part of this paper, we introduce a fuzzy model for LOB's order execution. As Rickard and Torre [16] point out, this allows for the modeling of an additional source of uncertainty, relating to 'the desirability of accomplishing a trade'.

Our model incorporates other sources of uncertainty as well. Of course, random changes in the fundamental value of the traded security affect the best bid (and ask) price. This effect is captured by assuming that such changes are driven by a Wiener process, as is common in the financial literature. We also allow for a non-zero drift in the price process, while the existing literature on optimal trade execution usually adopts a martingale for the price process. Furthermore, in contrast to most literature on trading with limit orders, we incorporate the effect of the bid-ask spread on the trader's strategy as well. This is also a new feature of the model, which requires a modeling of both sides of the limit order book. Finally, we model the effect of the uncertainty in the order execution parameter on the trader's decisions.

The base model is presented in Section 2; Section 3 extends the model to incorporate several realistic features; Section 4 introduces a fuzzy version of the model and offers some numerical examples.

2. The base model

In this section, the focus is on the problem of optimal purchase of a block of shares throughout an LOB. Consider an agent who needs to buy $\bar{X}$ shares within a time horizon T and submits limit orders of size x at the times $t_0, \dots, t_{N-1}$. Let $t_N = T$. Let $\hat{S}_{t_k}$ denote the limit price of the buy order that is posted at time t_k. In the sequel, it will be convenient to express the posted quote $\hat{S}_t$ in terms of its distance, δ_t, from the best bid price of the stock, S_t, which is assumed to change according to Brownian motion, i.e. $S_t = S_0 + \mu t + \sigma W_t$, where W_t is a Wiener process with respect to a given filtration $(F_t)_{t\geq 0}$. Throughout the paper, the conditional expectations E_t will be taken with respect to F_t, that is, E_t is a short notation for $E[.\,|\,F_t]$.

At any trading time, t_k, the agent observes the reference price $S_{t_k}$and decides the level of the quote $\delta_k = S_{t_k} - \hat{S}_{t_k}$ to submit. The sequence $(\delta_k)_{k=0,\dots,N-1}$ is chosen in order to optimize the agent's objective function, as explained below. Note that an order with a large and positive δ_k will incur a low probability of being filled, while $\delta_k < 0$ corresponds to an order in the market or a marketable one (Harris [10], for a precise definition), which is likely to be executed against incoming market orders. We assume that $(\delta_k)_{k=0,\dots,N-1}$ are adapted to the reference filtration.

As the main risk faced by the agent is the non-execution of his placed orders, we employ an indicator operator, I_j, taking value 1 if the order placed at time t_j is filled, and 0 in the opposite case. Let us restrict ourselves to the situation where an order submitted at time t_j is either executed or cancelled within the time t_{j+1}. The random variable I_j is independent of the future movements of the reference stock price. Furthermore, as motivated in the Introduction, it is assumed that the probability that a limit order δ_k gets executed is proportional to $\exp(-\lambda\delta_k)$, where the parameter λ depends on the trading environment, i.e. the LOB where orders are posted. Let $\Lambda(\delta)$ denote the conditional probability of execution.

Then the stock inventory in the time interval $[t_k, t_{k+1}]$, $k = 0, \dots, N-1$, can be written as

$$X_k = x\sum_{j=0}^{k} I_j.$$

Under a zero interest rate, the final wealth (at time t_N) is:

$$R_N = R_0 + x\sum_{k=0}^{N-1} I_k(\delta_k + S_{t_N} - S_{t_k})$$

where R_0 is the initial wealth, and the expected value of the terminal wealth is:

$$E_0[R_N] = R_0 + xE_0\left[\sum_{k=0}^{N-1}(\mu(t_N - t_k) + \delta_k)\Lambda(\delta_k)\right] \tag{1}$$

where $\Lambda(\delta_k)$ is the probability of order execution, conditional on δ_k, as defined above.

If the agent has missed his target at the terminal trade time, for example the total number of shares bought within the trading horizon is less than $\bar{X}$, then he incurs a penalty cost. An obvious interpretation is that he has to complete the target inventory by buying (or selling) the remaining shares at the market price, and so he incurs a quadratic transaction cost of the form $\frac{\ell}{2}(\bar{X} - X_{N-1})^2$. The parameter ℓ is a measure of the illiquidity of the trading environment and a quadratic shape for the cost due to market impact is widely adopted in the literature (See Almgren and Chriss [3] and its offspring in the related literature on optimal trading under market impact). Such a framework has been adopted in a hedging problem in Agliardi and Gençay [2].

The agent's optimization problem takes the form:

$$\sup_{\delta_0,\dots\delta_{N-1}} E_0\left[x\sum_{k=0}^{N-1}(\mu(t_N - t_k) + \delta_k)\Lambda_k - \frac{\ell}{2}(\bar{X} - X_{N-1})^2\right] \tag{2}$$

where $X_k = X_{k-1} + xI_k$, $k = 0, \dots, N-1$, and $X_{-1} = 0$.

In the following Proposition the optimization problem is solved.