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An extension of the Noether theorem: Accompanying equations possessing conservation laws

V.A. Dorodnitsyn [a], N.H. Ibragimov [b,c,*]

[a] Keldysh Institute of Applied Mathematics, Russian Acad. Sci., Moscow, Russia
[b] Laboratory "Group Analysis of Mathematical Models in Natural and Engineering Sciences", Ufa State Aviation Technical University, Ufa, Russia
[c] Research Centre ALGA: Advances in Lie Group Analysis, Department of Mathematics and Science, Blekinge Institute of Technology, Karlskrona, Sweden

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ABSTRACT

It is shown that the Noether theorem can be extended for some equations associated (accompanying) with Euler–Lagrange equation. Each symmetry of Lagrangian yields a class of accompanying equations possessing conservation law (first integral).

The generalization is done for canonical Hamiltonian equations as well.

1. Introduction

The well-known Noether's theorem [1] states that if the variational integral is invariant with respect to a one-parameter group of transformations then a certain formula provides a conservation law for the corresponding Euler–Lagrange equation. Thus, according to Noether's theorem, the invariance of the variational integral is a *sufficient condition* for existence of the conservation law. It has been proved in [2] that the *necessary and sufficient condition* for existence of this conservation law is the invariance of the value of the variational integral *on the solutions of the Euler–Lagrange equation.* The latter result establishes a one-to-one correspondence between a certain invariance of the variational integral and conservation laws.

The simplest way to prove the above statements is based on the following operator identity [3] (for the sake of simplicity we write it for second-order Euler–Lagrange equations):

$$\xi^i_\alpha \frac{\partial L}{\partial x^i} + \eta^k_\alpha \frac{\partial L}{\partial u^k} + [D_i(\eta^k_\alpha) - u^k_j D_i(\xi^j_\alpha)] \frac{\partial L}{\partial u^k_i} + L D_i(\xi^i_\alpha) = (\eta^k_\alpha - u^k_i \xi^i_\alpha) \frac{\delta L}{\delta u^k} + D_i\left((\eta^k_\alpha - u^k_i \xi^i_\alpha) \frac{\partial L}{\partial u^k_i} + L\xi^i_\alpha \right),$$

$$\alpha = 1, 2, \ldots, r. \tag{1.1}$$

Here $x = (x^1, \ldots, x^n)$ are the independent variables, $u = (u^1, \ldots, u^m)$ are the dependent variables with the first-order derivatives $u_{(1)} = \{u^k_i\}$, where

$$u^k_i = \frac{\partial u^k}{\partial x^i}.$$

* Corresponding author at: Research Centre ALGA: Advances in Lie Group Analysis, Department of Mathematics and Science, Blekinge Institute of Technology, Karlskrona, Sweden. Tel.: +46 455 385450; fax: +46 455 385460.

E-mail addresses: dorod2007@gmail.com (V.A. Dorodnitsyn), nib@bth.se, nailhib@gmail.com (N.H. Ibragimov).

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The variational integral with the Lagrangian L is invariant with respect to an r-parameter group of transformations generated by the operators

$$X_\alpha = \xi^i_\alpha \frac{\partial}{\partial x^i} + \eta^k_\alpha \frac{\partial}{\partial u^k},$$

if and only if the left-hand side of the identity (1.1) vanishes,

$$\xi^i_\alpha \frac{\partial L}{\partial x_i} + \eta^k_\alpha \frac{\partial L}{\partial u^k} + [D_i(\eta^k_\alpha) - u^k_j D_i(\xi^j_\alpha)] \frac{\partial L}{\partial u^k_i} + L D_i(\xi^i_\alpha) = 0. \tag{1.2}$$

2. Accompanying equations

The crucial observation for our extension of Noether's theorem is that the left-hand side of the identity (1.1) depends on first-order derivatives, whereas each summand in the right-hand side contains second order derivatives, which cancel each other upon summation. The situation is the same for higher-order Lagrangians L.

In view of this observation we suggest to modify the right-hand side of the identity (1.1) as follows:

$$\xi^i_\alpha \frac{\partial L}{\partial x^i} + \eta^k_\alpha \frac{\partial L}{\partial u^k} + [D_i(\eta^k_\alpha) - u^k_j D_i(\xi^j_\alpha)] \frac{\partial L}{\partial u^k_i} + L D_i(\xi^i_\alpha) = (\eta^k_\alpha - u^k_i \xi^i_\alpha)\left(\frac{\delta L}{\delta u^k} + \frac{D_i(B^i)}{(\eta^k_\alpha - u^k_i \xi^i_\alpha)}\right) + D_i\left((\eta^k_\alpha - u^k_i \xi^i_\alpha)\frac{\partial L}{\partial u^k_i} + L\xi^i_\alpha - B^i\right), \tag{2.1}$$

where $B^i = B^i(x^i, u^k, u^k_i, \ldots)$ are arbitrary smooth functions of a finite number of arguments.

Definition 1. The equations

$$\frac{\delta L}{\delta u^k} + \frac{D_i(B^i)}{(\eta^k_\alpha - u^k_i \xi^i_\alpha)} = 0, \quad \alpha = 1, 2, \ldots, r \tag{2.2}$$

are called *accompanying equations* for the Euler–Lagrange equations

$$\frac{\delta L}{\delta u^k} = 0, \quad k = 1, \ldots, m. \tag{2.3}$$

Proposition 1. *Let the variational integral with the Lagrangian L be invariant with respect to an r-parameter group of transformations. Then the accompanying Eqs. (2.2) possess the following conservation laws:*

$$D_i\left((\eta^k_\alpha - u^k_i \xi^i_\alpha)\frac{\partial L}{\partial u^k_i} + L\xi^i_\alpha - B^i\right) = 0, \quad \alpha = 1, 2, \ldots, r. \tag{2.4}$$

Proof. This property follows from the identity (2.1). □

Remark 1. According to Bessel-Hagen (see [4], the beginning of Section 1), Noether told him that in her theorem on conservation laws the invariance condition (1.2) can be replaced with the divergence condition

$$\xi^i_\alpha \frac{\partial L}{\partial x_i} + \eta^k_\alpha \frac{\partial L}{\partial u^k} + [D_i(\eta^k_\alpha) - u^k_j D_i(\xi^j_\alpha)] \frac{\partial L}{\partial u^k_i} + L D_i(\xi^i_\alpha) = D_i(P^i), \tag{2.5}$$

where $P^i = P^i(x^i, u^k, u^k_i, \ldots)$ are any smooth functions of a finite number of arguments. In this case the accompanying Eq. (2.2) have the following conservation laws:

$$D_i\left((\eta^k_\alpha - u^k_i \xi^i_\alpha)\frac{\partial L}{\partial u^k_i} + L\xi^i_\alpha - B^i - P^i\right) = 0, \quad \alpha = 1, 2, \ldots, r. \tag{2.6}$$

3. Application to second-order ODEs

We begin with the second-order ODEs possessing Lie point symmetries. Lie solved the problem of group classification of these equations [5] (see also [6], Table 9). The classification result is given in Table 1.