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Group classification of ODE $y''' = F(x, y, y')$

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ABSTRACT

In his extensive work of 1884 on the group classification of ordinary differential equations Lie performed, inter alia, the group classification of the particular type of the second-order equations $y'' = F(x, y)$. In the present paper we extend Lie's classification to the third-order equations $y''' = F(x, y, y')$.

1. Introduction

Recall that Lie solved the problem of group classification of the second-order ordinary differential equations (see, e.g. [1])

$$y'' = f(x, y, y') \tag{1.1}$$

and showed that the Eqs. (1.1) can admit symmetry Lie algebras L_r whose dimensions assume precisely the values 1, 2, 3, and 8.

Note that an arbitrary invertible change of variables

$$\bar{x} = \phi(x, y), \quad \bar{y} = \psi(x, y), \quad \phi_x \psi_y - \phi_y \psi_x \neq 0, \tag{1.2}$$

maps any Eq. (1.1) into an equation of the same form. In other words, Eq. (1.2) provides the general group of point wise equivalence transformations of Eq. (1.1) with arbitrary $f(x, y, y')$. Lie's group classification is made up to equivalence transformations (1.2). The classification result is given in Table 1.

In 1884 Lie performed the group classification (see [2], §2, pp. 440–446) of the particular type of Eqs. (1.1), namely of the equations

$$y'' = F(x, y). \tag{1.3}$$

Lie demonstrated by simple calculations that the general group of point wise equivalence transformations of Eq. (1.3) is given by the transformations

$$\bar{x} = \phi(x), \quad \bar{y} = A\sqrt{\phi'(x)}\, y + \beta(x), \tag{1.4}$$

where $A \neq 0$ is any constant, $\beta(x)$ is an arbitrary function, and $\phi(x)$ is an arbitrary function such that $\phi'(x) \neq 0$.

Lie's group classification of Eqs. (1.3) up to equivalence transformations (1.4) is summarized in Table 2.

The calculations show (see [3], Sections 2.1 and 2.2) that the transformation (1.4) maps Eq. (1.3) into the equation

$$\bar{y}'' = \bar{F}(\bar{x}, \bar{y})$$

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Table 1
Lie's group classification of equations $y'' = f(x, y, y')$.

L_r	Basis of L_r	Equation
L_1	$X_1 = \frac{\partial}{\partial x}$	$y'' = f(y, y')$
L_2	$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}$	$y'' = f(y')$
	$X_1 = \frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$	$y'' = \frac{1}{x}f(y')$
L_3	$X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad X_3 = x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}$	$y'' = 2\frac{y' + Cy'^{3/2} + y'^2}{y - x}$
	$X_1 = \frac{\partial}{\partial x}, \quad X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad X_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$	$y'' = Cy^{-3}$
	$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial x} + (x + y)\frac{\partial}{\partial y}$	$y'' = Ce^{-y'}$
	$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial x} + ky\frac{\partial}{\partial y}$	$y'' = Cy'^{\frac{k-2}{k-1}}, k \neq 0, \frac{1}{2}, 1, 2$
L_8	$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial y}, \quad X_4 = x\frac{\partial}{\partial x}, \quad X_5 = y\frac{\partial}{\partial x}, X_6 = y\frac{\partial}{\partial y}, \quad X_7 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \quad X_8 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}$	$y'' = 0$

Table 2
Lie's group classification of equations $y'' = F(x, y)$.

L_r	Basis of L_r	Equation
L_1	$X_1 = \frac{\partial}{\partial x}$	$y'' = F(y)$
	$X_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$	$y'' = y\Omega(ye^{-x})$
L_2	$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}$	$y'' = e^y$
	$X_1 = \frac{\partial}{\partial x}, \quad X_2 = (m-1)x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}$	$y'' = y^m$
L_3	See Table 1	$y'' = Cy^{-3}$
L_8	See Table 1	$y'' = 0$

with

$$\overline{F} = \frac{A}{(\phi')^{3/2}}F + A\left[\frac{\phi'''}{2(\phi')^{5/2}} - \frac{3(\varphi'')^2}{4(\phi')^{7/2}}\right]y + \frac{\beta''}{(\phi')^2} - \frac{\beta'\phi''}{(\phi')^3} \tag{1.5}$$

and that the Lie algebra of the equivalence transformations 1.4,1.5 is spanned by the operators

$$Y_0 = y\frac{\partial}{\partial y} + F\frac{\partial}{\partial F}, \quad Y_\beta = \beta(x)\frac{\partial}{\partial y} + \beta''(x)\frac{\partial}{\partial F},$$

$$Y_\xi = \xi(x)\frac{\partial}{\partial x} + \frac{y}{2}\xi'(x)\frac{\partial}{\partial y} + \left[\frac{y}{2}\xi'''(x) - \frac{3}{2}\xi'(x)F\right]\frac{\partial}{\partial F}. \tag{1.6}$$

Remark 1. The group classification of the equations $y'' = F(x, y)$ has been repeated later in [4], where the following equation and its symmetry $\overline{X}$:

$$\bar{y}'' = \frac{1}{\bar{x}^2}g(\bar{y}), \quad \overline{X} = \bar{x}\frac{\partial}{\partial \bar{x}} \tag{1.7}$$

have appeared instead of Lie's equation and its symmetry X (see Table 2):

$$y'' = y\Omega(ye^{-x}), \quad X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{1.8}$$

However, these two equations belong to one and the same equivalence class, namely they are connected by the following equivalence transformation (1.4):

$$\bar{x} = e^{-2x}, \quad \bar{y} = e^{-x}y. \tag{1.9}$$

The transformation (1.9) can be easily found from the requirement that the transformation (1.4) maps the operator X from Eq. (1.8) into the operator that is proportional to the operator $\overline{X}$ from Eq. (1.7), namely

$$X(\phi(x))\frac{\partial}{\partial \bar{x}} + X\left(A\sqrt{\phi'(x)}y + \beta(x)\right)\frac{\partial}{\partial \bar{y}} = \lambda\overline{X} \equiv \lambda\phi(x)\frac{\partial}{\partial \bar{x}}.$$

Indeed, equating the coefficients of $\partial/\partial\bar{x}$ in both sides of the above equation one obtains

$$\frac{d\phi}{dx} = \lambda\phi,$$