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On the nonlinear self-adjointness and local conservation laws for a class of evolution equations unifying many models

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ABSTRACT

In this paper we consider a class of evolution equations up to fifth-order containing many arbitrary smooth functions from the point of view of nonlinear self-adjointness. The studied class includes many important equations modeling different phenomena. In particular, some of the considered equations were studied previously by other researchers from the point of view of quasi self-adjointness or strictly self-adjointness. Therefore we find new local conservation laws for these equations invoking the obtained results on nonlinearly self-adjointness and the conservation theorem proposed by Nail Ibragimov.

1. Introduction

This paper partially corresponds to a talk given during MOGRAN 15 – October 02–06, 2012, Kemer, Turkey, where the first author had the opportunity to discuss about some recent results [6,8,27] relating local conservation laws and evolution equations.

The established conservation laws were obtained using recent developments on conserved quantities due to Nail Ibragimov since the fundamental work [14]. In the present paper we not only generalize the results presented during the conference, but we also show that some recent contributions in this field can be derived from the developments presented here.

In [12] Ibragimov introduced the concept of self-adjoint equation. According to the proposed definition, self-adjointness can now be considered not only to linear equations, but also to arbitrary (and nonlinear!) equations or systems.

Later, in [14], it was shown a new way for finding conserved quantities associated with any symmetry of a given equation or system. In what follows, we restrict ourselves only to a single differential equation, although it would be possible to consider a more general case.

The obtained conserved vector constructed by the results established in [14] provides a nonlocal conservation law for the original equation. In fact, the components of the conserved current is given by the following:

Theorem 1. *Let*

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u}$$

be any symmetry (Lie point, Lie-Bäcklund, nonlocal symmetry) of a given differential equation

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$$F(x,u,u_{(1)},\ldots,u_{(s)})=0 \tag{1}$$

and

$$F^{*}(x,u,v,\ldots,u_{(s)},v_{(s)}):=\frac{\delta}{\delta u}\mathcal{L}=0, \tag{2}$$

where $\mathcal{L}=vF$ *is the formal Lagrangian, be the adjoint equation to equation* (2)*. Then the combined system* (1) *and* (2) *has the conservation law* $D_iC^i=0$*, where*

$$C^i=\xi^i\mathcal{L}+W\left[\frac{\partial\mathcal{L}}{\partial u_i}-D_j\left(\frac{\partial\mathcal{L}}{\partial u_{ij}}\right)+D_jD_k\frac{\partial\mathcal{L}}{\partial u_{ijk}}-\cdots\right]+D_j(W)\left[\frac{\partial\mathcal{L}}{\partial u_{ij}}-D_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}}\right)+\cdots\right]+D_jD_k(W)\left[\frac{\partial\mathcal{L}}{\partial u_{ijk}}-\cdots\right]+\cdots \tag{3}$$

and $W=\eta-\xi^iu_i$.

For further details, see [14]. Observe that the components C^i given by (3) depend not only on the local variables, but also from the new variable $v=v(x)$ and its derivatives. Therefore, such a considered vector provides a nonlocal conservation law to the original Eq. (1).

However, invoking the concept of self-adjoint equation introduced in [12] (and after renamed as strictly self-adjoint equation in [20]), it is possible to obtain certain special equations, called self-adjoint differential equations, which the Theorem 1, hereafter called *Ibragimov theorem*, provides a local conservation law for the original equations. The reason for this "phenomenon" is due to the following: a strictly self-adjoint equation possesses the property that under the change $v=u$, the adjoint equation becomes original to the first one. Therefore, system $F=0$ and $F^*=0$ is equivalent to only the original equation under this substitution on the nonlocal variable v. Then, into the components (3) one can eliminate the nonlocal variable v by substituting u instead of v.

This interesting idea allowed many people to obtain local conserved quantities employing Theorem 1 for equations without variational structure, see [3–5,17,19] and references therein for some examples.

Since then, the concept of strictly self-adjoint equation has been generalised, see [15,9,18,20]. Although these concepts are very useful for finding conserved quantities from the Ibragimov theorem, throughout the years a lot of papers have been dedicated not only to find conservation laws, but also for classifying equations under some kind of self-adjointness, see [3–8,10,16,17,19,24] and references therein.

In this paper we find necessary and sufficient conditions in order to the following class of equations

$$u_t+f(t,u)u_{xxxxx}+r(t,u)u_{xxx}+g(t,u)u_xu_{xx}+h(t,u)u_x^3+a(t,u)u_x+b(t,u)=0, \tag{4}$$

be nonlinearly self-adjoint, that is, we consider the problem on nonlinear self-adjoint classification of (4), that is done in the Section 3. In the following, we illustrate how to find local conserved quantities from a nonlinearly self-adjoint equation. In order to do this, in the Section 2 we revisit the basic tools of the *Ibragimov theory on conservation laws*.

2. Conservation laws and Ibragimov's developments

In this section we recall some basic concepts related to conservation laws and a general view on the developments introduced by Ibragimov. The diligent and interested reader is strongly encouraged to read the fundamental works [14,15,18,20]. See also [4–8].

The summation over the repeated indices is presupposed. All functions here are assumed to be smooth. If $x=(x^1,\ldots,x^n)$ are n independent variables and $u=(u^1,\ldots,u^p)$ are p dependent variables, then $u_{(k)}$ denotes the set of all kth derivatives of u. Moreover,

$$u^{\alpha}_{i_1\cdots i_j}=D_{i_1}\ldots D_{i_j}(u^{\alpha}),$$

where

$$D_i=\frac{\partial}{\partial x^i}+u_i^{\alpha}\frac{\partial}{\partial u^{\alpha}}+u_{ij}^{\alpha}\frac{\partial}{\partial u_j^{\alpha}}+\cdots$$

are the total derivative operators and

$$\frac{\delta}{\delta u^{\alpha}}=\frac{\partial}{\partial u^{\alpha}}+\sum_{j=0}^{\infty}(-1)^jD_{i_1}\cdots D_{i_j}\frac{\partial}{\partial u^{\alpha}_{i_1\cdots i_j}}$$

are the Euler–Lagrange operators.

Definition 1. A locally analytic function of a finite number of the variables x,u and u derivatives is called a differential function. The highest order of derivatives appearing in the differential function is called the order of this function. The vector space of all differential functions of finite order is denoted by $\mathcal{A}$.

Given a differential equation $F(x,u,u_{(1)},\ldots,u_{(m)})=0$, a conservation law for this equation is a vanishing divergence on the solutions of the equation, that is,