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Group classification and conservation laws of nonlinear filtration equation with a small parameter

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ABSTRACT

Group classification of the perturbed nonlinear filtration equation is performed assuming that the perturbation is an arbitrary function of the dependent variable. The nonlinear self-adjointness of the equation under consideration is investigated. Using these results, the approximate conservation laws are constructed.

1. Introduction

We consider the nonlinear filtration equation with a small parameter:

$$u_t \approx k(u_x)u_{xx} + \varepsilon f(u). \tag{1.1}$$

In particular, this equation describes a pressure distribution in a porous medium. In this paper we make the group classification of Eq. (1.1) and search approximate conservation laws for this equation using its symmetries.

Throughout paper the approximate equation

$$f(x) \approx g(x)$$

means that

$$f(x,\varepsilon) = g(x,\varepsilon) + o(\varepsilon).$$

2. Group classification

The generator of an approximate transformation group admitted by Eq. (1.1) will be written in the form (see [1])

$$X = \xi^1(t,x,u,\varepsilon)\frac{\partial}{\partial t} + \xi^2(t,x,u,\varepsilon)\frac{\partial}{\partial x} + \eta(t,x,u,\varepsilon)\frac{\partial}{\partial u}, \tag{2.2}$$

where

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$$\xi^i(t,x,u,\varepsilon) \approx \xi^i_{(0)}(t,x,u) + \varepsilon\xi^i_{(1)}(t,x,u), \quad i = 1,2,$$
$$\eta(t,x,u,\varepsilon) \approx \eta_{(0)}(t,x,u) + \varepsilon\eta_{(1)}(t,x,u). \tag{2.3}$$

The determining equation for the operator (2.2) has the form

$$(\varepsilon f'(u)\eta_{(0)} - \zeta_{1(0)} - \varepsilon\zeta_{1(1)} + k'(u_x)u_{xx}(\zeta_{2(0)} + \varepsilon\zeta_{2(1)}) + k(u_x)(\zeta_{22(0)} + \varepsilon\zeta_{22(1)}))|_{(1.1)} \approx 0, \tag{2.4}$$

where $\zeta_{i(j)},\ \zeta_{22(j)}$ are obtained by the usual prolongation formulae:

$$\zeta_{i(j)} = D_i(\eta_{(j)}) - u_t D_i(\xi^1_{(j)}) - u_x D_i(\xi^2_{(j)}), \quad i = 1,2, \quad j = 0,1,$$
$$\zeta_{22(j)} = D_2(\zeta_{2(j)}) - u_{tx} D_2(\xi^1_{(j)}) - u_{xx} D_2(\xi^2_{(j)}).$$

After replacing u_t by $k(u_x)u_{xx} + \varepsilon f(u)$ in (2.4) and splitting this equation by variable ε we obtain the following system (see [1])

$$\zeta_{1(0)} - \zeta_{2(0)}k'(u_x)u_{xx} - k(u_x)\zeta_{22(0)} = 0, \tag{2.5}$$
$$\zeta_{1(1)} - f'(u)\eta_{(0)} - \zeta_{2(1)}k'(u_x)u_{xx} - k(u_x)\zeta_{22(1)} = 0. \tag{2.6}$$

Eq. (2.5) is the determining equation for the nonlinear filtration equation

$$u_t = k(u_x)u_{xx}. \tag{2.7}$$

We will use the known result of the group classification of Eq. (2.7) (see, e.g. [2]). This result is summarized in Table 1. Eq. (2.6) gives the following system of equations for $\xi^1_{(1)},\ \xi^2_{(1)},\ \eta_{(1)}$:

$$\begin{aligned}\xi^1_{(1)x} + u_x\xi^1_{(1)u} = 0, k(u_x)\left[-\xi^1_{(1)t} + 2\xi^2_{(1)x} + 2u_x\xi^2_{(1)u}\right] \\ = k'(u_x)\left[\eta_{(1)x} + (\eta_{(1)u} - \xi^2_{(1)x})u_x - \xi^2_{(1)u}u_x^2)\right], f(u)(\eta_{(0)u} - \xi^1_{(0)t}) - f'(u)\eta_{(0)} + \eta_{(1)t} - u_x\xi^2_{(1)t} \\ = k(u_x)\left[\eta_{(1)xx} + (2\eta_{(1)xu} - \xi^2_{(1)xx})u_x + (\eta_{(1)uu} - 2\xi^2_{(1)xu})u_x^2 - \xi^2_{(1)uu}u_x^3\right],\end{aligned} \tag{2.8}$$

where $\xi^1_{(0)},\ \xi^2_{(0)},\ \eta_{(0)}$ are known functions that depend on the form of $k(u_x)$ (see [2]). In the case of an arbitrary function $k(u_x)$, system (2.8) splits into the system of equations

$$\begin{aligned}&\xi^1_{(1)x} = 0, \quad \xi^1_{(1)u} = 0, \quad 2\xi^2_{(1)x} - \xi^1_{(1)t} = 0, \quad \xi^2_{(1)u} = 0, \quad \xi^2_{(1)t} = 0,\\ &\eta_{(1)x} = 0, \quad \eta_{(1)u} - \xi^2_{(1)x} = 0, \quad \eta_{(1)uu} - 2\xi^2_{(1)xu} = 0,\\ &2\eta_{(1)xu} - \xi^2_{(1)xx} = 0, \quad f(u)(\eta_{(0)u} - \xi^1_{(0)t}) - f'(u)\eta_{(0)} + \eta_{(1)t} = 0.\end{aligned} \tag{2.9}$$

This system has the following solution:

$$\xi^1_{(1)} = 2A_1t + A_2, \quad \xi^2_{(1)} = A_1x + A_3, \quad \eta_{(1)} = A_1u + \alpha(t),$$

where A_i are constants, $i = 1,\ 2,\ 3$, and the function $\alpha(t)$ satisfies the last equation of system (2.9)

$$\alpha'(t) = (C_1u + C_2)f'(u) + C_1f(u).$$

Note that the function α here depends only on t, while the right-hand side of this formula depends only on u. Therefore this equality is possible only if $\alpha'(t)$ is a constant. Let it be A_4.

$$(C_1u + C_2)f'(u) + C_1f(u) = A_4. \tag{2.10}$$

If $f(u)$ is an arbitrary function, then coefficients $C_1u + C_2$ and C_1 of Eq. (2.10) vanish, i.e $C_1u + C_2 = 0$ and $C_2 = 0$. It follows that $C_1 = 0,\ C_2 = 0$, and we obtain the six-dimensional Lie algebra spanned by the following generators:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \varepsilon\frac{\partial}{\partial t}, \quad X_4 = \varepsilon\frac{\partial}{\partial x}, \tag{2.11}$$
$$X_5 = \varepsilon\frac{\partial}{\partial u}, \quad X_6 = \varepsilon 2t\frac{\partial}{\partial t} + \varepsilon x\frac{\partial}{\partial x} + \varepsilon u\frac{\partial}{\partial u}.$$

Table 1
Group classification of $u_t = k(u_x)u_{xx}$.

$k(u_x)$	Generators
Arbitrary	$X_1 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u},\ X_2 = \frac{\partial}{\partial u},\ X_3 = \frac{\partial}{\partial t},\ X_4 = \frac{\partial}{\partial x}$
e^{u_x}	$X_5 = t\frac{\partial}{\partial t} - x\frac{\partial}{\partial u}$
$u_x^{\sigma-1}$, where $\sigma \geqslant 0,\ \sigma \neq 1$	$X_5 = (1-\sigma)t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$
$\frac{1}{u_x^2+1}e^{\sigma \arctan u_x}$, where $\sigma \geqslant 0$	$X_5 = \sigma t\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} - x\frac{\partial}{\partial u}$