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Conservations laws for a porous medium equation through nonclassical generators

M.L. Gandarias

Departamento de Matemáticas, Universidad de Cádiz, 11510 Puerto Real, Cádiz, Spain

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ABSTRACT

In Ibragimov (2007) [13] a general theorem on conservation laws was proved. In Gandarias (2011) and Ibragimov (2011) [7,15] the concepts of self-adjoint and quasi self-adjoint equations were generalized and the definitions of weak self-adjoint equations and nonlinearly self-adjoint equations were introduced. In this paper, we find the subclasses of nonlinearly self-adjoint porous medium equations. By using the property of nonlinear self-adjointness, we construct some conservation laws associated with classical and nonclassical generators of the differential equation.

1. Introduction

For decades, researchers have been interested in the description of evolution processes where diffusion is combined with other effects, notably reaction, absorption and convection. While more often that not the spatial-dependent factors are believed to be constant, there is no fundamental reason to believe so. In fact, allowing their spatial dependence enables one to incorporate additional factors into the study which may play an important role. For instance, in a porous medium this may account for intrinsic factors, like medium contamination with another material. Also, in plasma, this may express the impact that solid impurities arising from the walls have on the enhancement of the radiation channel. The model equation to be considered here is the one-dimensional evolution equation involving diffusion and convection

$$u_t = (u^n)_{xx} + f(x)u^s u_x. \tag{1.1}$$

In some previous works [6,9], we study Eq. (1.1) from the point of view of the theory of symmetry reductions in partial differential equations. We obtain the classical and nonclassical symmetries admitted by (1.1), we list the different choices for functions $f(x)$ and constants n and s, for which Eq. (1.1) admits classical and nonclassical reductions. Then, we use the transformations groups to reduce the equations to ordinary differential equations.

In [1,2] Anco and Bluman gave a treatment of a direct conservation law method for partial differential equations expressed in a standard Cauchy–Kovaleskaya form. The nontrivial conservation laws are characterized by a multiplier λ with no dependence on u_t. In [19] Kara and Mahomed showed how to construct conservation laws of Euler–Lagrange type equations via Noether type symmetry operators associated with partial Lagrangians.

In [13], (see also [12]) a general theorem was proved in order to get conservation laws for arbitrary differential equations. This theorem allows us to find for any differential equation with known Lie, Lie-Bäcklund or nonlocal symmetries, the associated conservation laws independently of the existence of classical Lagrangians. Many equations having remarkable symmetry properties and physical significance are not self-adjoint. The notion of self-adjoint equations has been extended and

E-mail address: marialuz.gandarias@uca.es

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the concepts of quasi self-adjoint equations, weak self-adjoint and nonlinearly self-adjoint equations, have been introduced in [14,7,15]. Recently, many works have been done in this direction to get conservation laws associated to classical symmetries for some nonlinear differential equations [5,3,4,17,18].

In some previous papers [7,8], we have determined the subclasses of weak self-adjoint porous medium equations and we have constructed some conservation laws associated with classical symmetries of weak self-adjoint differential equations.

The aim of this paper is to determine, for Eq. (1.1), the subclasses of equations which are nonlinearly self-adjoint. We will also determine, by using the notation and techniques of [13], some non-trivial conservation laws for Eq. (1.1) associated to classical and nonclassical generators. As far as we know, it is the first time in which the general theorem on conservation laws proved in [13] has been used to find conservation laws associated to nonclassical generators.

2. Weak and nonlinearly self-adjoint equations

Definition 1. Consider a sth-order partial differential equation

$$F(x,u,u_{(1)},\ldots,u_{(s)})=0, \tag{2.1}$$

with independent variables $x=(x^1,\ldots,x^n)$ and a dependent variable u, where $u_{(1)}=\{u_i\}, u_{(2)}=\{u_{ij}\},\ldots$ denote the sets of the partial derivatives of the first, second, etc. orders, $u_i=\partial u/\partial x^i$, $u_{ij}=\partial^2 u/\partial x^i\partial x^j$.

The adjoint equation to (2.1) is

$$F^*(x,u,v,u_{(1)},v_{(1)},\ldots,u_{(s)},v_{(s)})=0, \tag{2.2}$$

with

$$F^*(x,u,v,u_{(1)},v_{(1)},\ldots,u_{(s)},v_{(s)})=\frac{\delta(vF)}{\delta u}, \tag{2.3}$$

where

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}+\sum_{s=1}^{\infty}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u_{i_1\cdots i_s}}, \tag{2.4}$$

denotes the variational derivative (the Euler–Lagrange operator), and v is a new dependent variable.

Here

$$D_i=\frac{\partial}{\partial x^i}+u_i\frac{\partial}{\partial u}+u_{ij}\frac{\partial}{\partial u_j}+\cdots$$

are the total differentiations.

Definition 2. Eq. (2.1) is said to be *weak self-adjoint* if the equation obtained from the adjoint equation (2.2) by the substitution $v=h(x,u)$, with a certain function $h(x,u)$ such that $h_x\neq 0$ and $h_u\neq 0$ is identical to the original equation.

Definition 3. Eq. (2.1) is said to be *nonlinearly self-adjoint* if the equation obtained from the adjoint equation (2.2) by the substitution $v=h(x,u)$, with a certain function $h(x,u)$ such that $h(x,u)\neq 0$ is identical to the original equation.

We remark that it is possible to find substitutions $v=h(x,t,u,u_x,\ldots)$ depending on the derivatives, see [16,10] for further discussions and examples.

Following [12], applying the Euler–Lagrange operator to the formal Lagrangian $L=vF$ we obtain that the adjoint equation to (1.1) reads:

$$F^*=\tfrac{\delta}{\delta u}[v(u_t-(u^n)_{xx}-f(x)u^s u_x)]=-nu^{n-1}v_{xx}+fu^s v_x-v_t+f_x u^s v \tag{2.5}$$

Now we look for a substitution $v=h(x,u)$ such that Eq. (1.1) becomes nonlinearly self-adjoint. For this purpose we substitute $v=h(x,u)$ into (2.5)

$$F^*=-u^{n-2}(h_u nu)u_{xx}-u^{n-3}(h_{uu}nu^2)(u_x)^2+(fh_u u^s-2h_{ux}nu^{n-1})u_x-h_u u_t+(fh_x+f_x h)u^s-h_{xx}nu^{n-1}.$$

Now we assume that

$$F^*|_{v=h(x,u)}=\lambda[u_t-(u^n)_{xx}-f(x)u^s u_x] \tag{2.6}$$

where λ is an undetermined coefficient. Condition (2.6) reads

$$u_x(fu^s\lambda+fh_u u^s-2h_{ux}nu^{n-1})+u^{n-3}(u_x)^2(n^2u\lambda-nu\lambda-h_{uu}nu^2)+u^{n-2}u_{xx}(nu\lambda-h_u nu)-u_t(\lambda+h_u)$$
$$+(fh_x+f_x h)u^s-h_{xx}nu^{n-1}=0. \tag{2.7}$$