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Conservation laws for two-phase filtration models

V.A. Baikov [a], N.H. Ibragimov [a,b,*], I.S. Zheltova [a], A.A. Yakovlev [a]

[a] Laboratory "Group Analysis of Mathematical Models in Natural and Engineering Sciences", Ufa State Aviation Technical University, Ufa, Russia
[b] Research Centre ALGA: Advances in Lie Group Analysis, Department of Mathematics and Science, Blekinge Institute of Technology, Karlskrona, Sweden

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ABSTRACT

The paper is devoted to investigation of group properties of a one-dimensional model of two-phase filtration in porous medium. Along with the general model, some of its particular cases widely used in oil-field development are discussed. The Buckley–Leverett model is considered in detail as a particular case of the one-dimensional filtration model. This model is constructed under the assumption that filtration is one-dimensional and horizontally directed, the porous medium is homogeneous and incompressible, the filtering fluids are also incompressible. The model of "chromatic fluid" filtration is also investigated. New conservation laws and particular solutions are constructed using symmetries and nonlinear self-adjointness of the system of equations.

1. Introduction

At present development of perspective undrilled fields (or at the stage of early development) is planned to be carried out in rim zones with degraded filtering reservoir properties. Beds in these zones are characterized by gas permeabilities $5 \cdot 10^{-4} < K < 5 \cdot 10^{-3}$ mcm^2 (1 mcm^2 = darcy) and characteristic pore sizes $d < A \cdot 10^{-7}$ m, where A is a value of the order of several units.

In practice, simulation of fluid filtration in porous medium is often used for effective control over the development of oil and gas fields. It should be mentioned that the fundamental law of filtration is the Darcy law, which was established already in the middle of the XIX century. A considerable number of works by Russian and world experts is devoted to verification and investigation of applicability of Darcy's law. In the course of these studies it has been shown that we can single out the upper and lower limits of applicability of Darcy's law. The upper limit has been well studied: the deviation is determined by a number of reasons related to occurrence of inertial forces at high filtration rates. During the last decades, the lower limit has been made conditional on a non-Newtonian behavior of oil. However, due to the increased interest in low-permeability formations, scientific publications of recent years are devoted to non-linear filtration (deviation from the linear Darcy law) in low permeability reservoirs.

Since the physical and mathematical model of filtration (represented by a system of nonlinear differential equations) is complex, there is no analytical solution in a general form. Numerical calculations are used to solve the system of differential equations. However, explicit analytical solutions are necessary for some specific problems arising in modeling an oil reservoir, including assessment of nonlinearity effects of the filtration law in simulation of field development, understanding of physical processes, estimation of development efficiency.

* Corresponding author at: Research Centre ALGA: Advances in Lie Group Analysis, Department of Mathematics and Science, Blekinge Institute of Technology, Karlskrona, Sweden. Tel.: +46 455 385450; fax: +46 455 385460.

E-mail addresses: baikov@ufanipi.ru (V.A. Baikov), nailhib@gmail.com (N.H. Ibragimov), zheltovais@gmail.com (I.S. Zheltova), yakovlevandrey@yandex.ru (A.A. Yakovlev).

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It is well known [4], that group analysis is a universal and efficient method for solving nonlinear differential equations analytically. Moreover, by recent works in group analysis [5] much prominence is given to construction of conservation laws, which are one of the main tools for constructing and investigating mathematical models and can be used to obtain some particular solutions.

The present paper provides investigation of a one-dimensional model of two-phase filtration in a general form together with its particular cases describing filtration of an incompressible fluid in an incompressible isotropic porous medium, as well as filtration of "colored" fluid. New conservation laws are constructed in the paper.

2. Two-phase filtration model

In this paper we use the model of a two-phase flow in porous media for simulation of displacing oil by water. The model is constructed with the assumption that water and oil do not blend, do not exchange masses, and do not change phases. Fluids in the reservoir are in thermodynamical equilibrium at a constant temperature. In what follows, the indices o, w denote oil and water phases, respectively. Equations of the model derived from the mass conservation equation and Darcy's law have the form [1–3]:

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\varphi(p_o,p_w)\frac{S}{B_w(p_w)}\right)+\nabla\left(\frac{KK_{rw}(S)}{\mu_w(p_w)B_w(p_w)}(\nabla p_w-\mathbf{g}\rho_w(p_w))\right)=q_w,\\
&\frac{\partial}{\partial t}\left(\varphi(p_o,p_w)\frac{1-S}{B_o(p_o)}\right)+\nabla\left(\frac{KK_{ro}(S)}{\mu_o(p_o)B_o(p_o)}(\nabla p_o-\mathbf{g}\rho_o(p_o))\right)=q_o,\\
&p_w=p_o+p_{cwo}(S),\quad K=K(\mathbf{x},p_o,p_w,\nabla p_o,\nabla p_w),
\end{aligned} \tag{2.1}$$

where $t, \mathbf{x}$ are independent variables, pressures of the oil and water phase $p_o(t,\mathbf{x}), p_w(t,\mathbf{x})$, and current water saturation $S(t,\mathbf{x})$ are unknown functions.

The vector $\mathbf{g}$ is defined as $\mathbf{g} = g\nabla z$, where g is a gravitational constant, and the axis z is supposed to be vertical.

Pressure and water saturation functions involved in the system (2.1) and describing properties of the porous medium, filtered fluids and their interaction are determined by experiment:

- Rock properties: K is a tensor of absolute permeability of the porous medium, φ is porosity of the medium;
- fluid properties: B_o, B_w are coefficients of volumetric oil and water expansion, ρ_o, ρ_w are oil and water densities, μ_o, μ_w are oil and water viscosities;

- interaction of rock and fluids: K_{ro}, K_{rw} are respective phase permeabilities for oil and water, p_{cwo} is a capillary preasure.

The functions $q_w(p_w,S,\mathbf{x}), q_o(p_o,S,\mathbf{x})$ in the right-hand side of the system of Eqs. (2.1) are sinks/sources of phases.

In what follows we consider filtration in a straight thin sample, represented by an isotropic porous medium, i.e. K is independent of $\mathbf{x}$. Let us assume that the cross section of the sample is small such that the pressure and saturation can be considered to be constant with respect to the the cross sections. Sources and sinks are absent. The pressure in the aqueous and oil phases is considered to be the same because the surface tension between the phases is small (capillary pressure can be neglected)

$$p_o(t,x)=p_w(t,x)=p(t,x).$$

The system of Eqs. (2.1) takes the form

$$\begin{aligned}
&\Lambda^1\equiv\left(\frac{\varphi(p)S}{B_w(p)}\right)_t+\left(\frac{K(p,p_x)K_{rw}(S)}{\mu_w(p)B_w(p)}(p_x-\tilde{g}\rho_w(p))\right)_x=0,\\
&\Lambda^2\equiv\left(\frac{\varphi(p)(1-S)}{B_o(p)}\right)_t+\left(\frac{K(p,p_x)K_{ro}(S)}{\mu_o(p)B_o(p)}(p_x-\tilde{g}\rho_o(p))\right)_x=0,
\end{aligned} \tag{2.2}$$

where $\tilde{g}$ is a projection of the vector $\mathbf{g}$ to the axis x.

3. Admitted transformation group and conservation laws

Let us consider basic notions of Lie group theory to be used in what follows.

Transformations

$$\begin{aligned}
&\bar{t}=g_1(t,x,p,S,a),\quad \bar{x}=g_2(t,x,p,S,a),\\
&\bar{p}=g_3(t,x,p,S,a),\quad \bar{S}=g_4(t,x,p,S,a)
\end{aligned} \tag{3.3}$$

of the variables t, x, p, S, depending on the continuous parameter a, are referred to as symmetry transformations of the system of Eqs. (2.2) if Eqs. (2.2) have the same form in the new variables $\bar{t}, \bar{x}, \bar{p}, \bar{S}$. A set of all such transformations composes a continuous group G. The symmetry group G is termed as an admitted group of the system (2.2).