Commun Nonlinear Sci Numer Simulat 19 (2014) 390–400

Contents lists available at SciVerse ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

An approximate solution method for ordinary fractional differential equations with the Riemann–Liouville fractional derivatives

S.Yu. Lukashchuk

Laboratory "Group Analysis of Mathematical Models in Natural and Engineering Sciences", Ufa State Aviation Technical University, 12 K. Marx str., Ufa 450000, Russia

ARTICLE INFO

Article history:
Available online 24 June 2013

Keywords:
Fractional differential equation
Two scales
Approximate solution

ABSTRACT

A new method is proposed to construct the approximate solutions of ordinary fractional differential equations with the Riemann–Liouville fractional derivatives. The method is based on the two scale technique. A fractional part of the order of the fractional derivative is considered as a small parameter ε, and two different scales x and x^{ε} are introduced. As a result, the fractional differential equation is reduced to a series of integer-order differential equations, all of that are linear, except may be first one. Two different approaches to initial conditions for this series of equations are discussed. Some examples illustrate the efficiency of the proposed method.

1. Introduction

Perturbation methods (see, e.g., [1,2]) are widely used for investigation the ordinary and partial differential equations modelling various phenomena. During last decade some classical perturbation methods were applied to the fractional differential equations with a small parameter [3–5].

A new perturbation technique called homotopy perturbation method (HPM) has been also proposed by He [6]. The main advantage of HPM is that it does not require a small parameter. This method and its different modifications have been successfully used for approximate solution of the various differential equations of fractional order (see, e.g., [7,8] and references therein).

In all methods mentioned above the fractional differential equations should be solved. Another perturbation technique can be developed for the fractional differential equations in which an order of the fractional derivative is close to an integer number. A difference between them can be considered as a small parameter ε. In this case a series of differential equations of integer (not fractional!) order should be solved to get an approximate solution of the fractional differential equation. Using such approach, [9] develop the ε-expansions for the fractional linear and nonlinear oscillators and the fractional Ginzburg–Landau equation, and show that these expansions are not uniform with respect to the independent variable. This formalism was extended by Tofighi and Golestani [10] who show that the uniform expansions can be constructed.

It is necessary to note that perturbation methods usually developed for the fractional differential equations with the Caputo fractional derivatives. Nevertheless, the equations with the Riemann–Liouville one are also widely used to describe the anomalous phenomena. The perturbation methods for such equations should take into account some features. Firstly, such equations are accompanied by the specific initial or boundary conditions which contain integrals or/and derivatives of fractional order. If a small parameter is extracted from the order of the fractional differentiation, then these initial (boundary)

E-mail address: lsu@mail.rb.ru

1007-5704/$ - see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.cnsns.2013.06.021

conditions also depend on this small parameter. Secondly, the solutions of such equations usually have a singularity in the initial (or boundary) points.

In this paper, a two scale perturbation method is proposed for ordinary fractional differential equations with the Riemann–Liouville fractional derivatives. The method permits to construct an approximate solution which has the same order of singularity in the initial point as an exact solution and therefore this approximate solution is valid in the neighborhood of the initial point.

2. Method description

Let us consider an ordinary fractional differential equation

$$D^{\alpha}y = f(x,y), \quad \alpha \in (0,1) \tag{2.1}$$

with the Riemann–Liouville fractional derivative

$$D^{\alpha}y = D(I^{1-\alpha}y)(x) \equiv \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x \frac{y(\xi)}{(x-\xi)^{\alpha}}d\xi \tag{2.2}$$

(here $I^{1-\alpha}y$ is the Riemann–Liouville fractional integral).

Eq. (2.1) is subject to the initial condition

$$(I^{1-\alpha}y)(0) = c, \quad c \in \mathbb{R}. \tag{2.3}$$

A fractional part of the order of fractional differentiation is considered as a small parameter in the proposed technique. Two different cases are distinguished: $\alpha = 1-\varepsilon$ for $\alpha > 1/2$ and $\alpha = \varepsilon$ for $\alpha < 1/2$. So, in both cases the small parameter $\varepsilon \in (0, 0.5)$. To illustrate the approach, the case $\alpha = 1-\varepsilon$ will be discussed in detail.

Suppose that for $x > 0$ the solution $y(x)$ of Eq. (2.1) is an analitic function. Then the following series representation of (2.2) is valid (see [11]):

$$D^{\alpha}y = \sum_{k=0}^{\infty}\binom{\alpha}{k}\frac{x^{k-\alpha}}{\Gamma(k+1-\alpha)}y^{(k)}(x). \tag{2.4}$$

Substituting in (2.4) $\alpha = 1-\varepsilon$, we obtain

$$D^{1-\varepsilon}y = \Gamma(2-\varepsilon)\frac{\sin(\pi\varepsilon)}{\pi}x^{\varepsilon}\sum_{k=0}^{\infty}\frac{(-1)^k x^{k-1}}{(1-k-\varepsilon)k!}y^{(k)}(x).$$

The multiplier x^{ε} can not be expanded in a convergent series of ε for all values of $x \in (0,\infty)$. Therefore, during the ε-expansion this term should be considered as a new independent variable. So, we have two scales: x and x^{ε}.

Note that for small values of x ($x \ll 1$) a new varible x^{ε} is a fast variable while x is a slow variable. For large values of x ($x \gg 1$) the variable's behaviour is reversed: now x is a fast and x^{ε} is a slow variable. Of cause, both variables have a closed behaviour in the neighborhood of the point $x = 1$.

It is known (see [12]) that if both $\lim_{x\to 0} I^{\alpha}y(x) = c$ and $\lim_{x\to 0} x^{\alpha}y(x)$ exist then

$$\lim_{x\to 0}(x^{\alpha}y(x)) = c/\Gamma(1-\alpha).$$

Therefore, condition (2.3) can be replaced by

$$\lim_{x\to 0}((x^{\varepsilon}y(x)) = c/\Gamma(1-\varepsilon). \tag{2.5}$$

We introduce a new dependent variable

$$z(x;\varepsilon) = x^{\varepsilon}y(x). \tag{2.6}$$

Contrary to $y(x)$, this new function has no singularity in the initial point $x = 0$.

Using Leibnitz rule (see, e.g., [11])

$$D^{\alpha}(f(x)g(x)) = \sum_{k=0}^{\infty}\binom{\alpha}{k}D^{\alpha-k}(f)D^k(g),$$

we find

$$D^{\alpha}y \equiv D^{1-\varepsilon}(x^{-\varepsilon}z) = \sum_{k=0}^{\infty}\binom{1-\varepsilon}{k}D^{1-\varepsilon-k}(x^{-\varepsilon})D^k(z) = \Gamma(1-\varepsilon)\Gamma(2-\varepsilon)\left[\Gamma(1+\varepsilon)\frac{dz}{dx} + \sum_{n=1}^{\infty}\frac{\varepsilon\Gamma(n+\varepsilon)}{n!(n+1)!}(-x)^n D_x^{n+1}(z)\right]. \tag{2.7}$$

Denote new scale by $x_1 = x^{\varepsilon}$. Then the derivative with respect to x is transformed according to