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A mathematical model for the admission process in intensive care units ☆

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ABSTRACT

A mathematical model is given for the admission process in Intensive Care Units (ICUs). It is shown that the model exhibits bistability for certain values of its parameters. In particular, it is observed that in a two-dimensional parameter space, two saddle-node bifurcation curves terminate at a single point of the cusp bifurcation, creating an enclosed region in which the model has one unstable and two stable states. It is shown that in the presence of bistability, variations in the value of parameters may lead to undesired outcomes in the admission process as the value of state variables abruptly changes. Using numerical simulations, it is also discussed how such outcomes can be avoided by appropriately adjusting the parameter values.

1. Introduction and problem statement

An Intensive Care Unit (ICU) serves as a place for monitoring and taking care of patients who are critically ill. It is quite usual for some patients to stay in ICUs for a long time; in other words, there are no specific time limits for receiving medical care in an ICU. Therefore, the need for efficient utilization of ICUs resources necessitates detailed investigations on admission, discharge and triage processes, in order to help hospitals take necessary measures to reduce costs and avoid undesired outcomes. A number of articles have studied ICUs patient flow [12,25], and its variability [19] by using algorithms, statistical techniques and computer simulations. By extending models introduced in [23,30], Costa et al. [4] studied mathematical models that can be used for making decisions about the number of beds required in a critical care unit. They used Classification and Regression Tree (CART) analysis to divide patients into several groups based on a chosen criterion such as length of stay. Many articles, in recent years, have proposed simulations and applied queuing models to study bed capacities management in hospitals; see [5,6,8,18]. Barado et al. [1] have developed a mathematical model based on the queuing theory to simulate daily bed occupancy in an ICU. Their computerized simulation model addresses the problem of approximating the number of beds needed in case where there is an increase in the number of patients in need of critical care. Other types of simulation modeling techniques for investigating problems in critical care medicine have been reviewed by Kreke et al. [13]. These techniques include Markov modeling [26], Mont Carlo simulation and discrete-event simulation (DES) [24]. They use a set of hypothetical input data to produce a set of expected outcomes. Litvak et al. [17] have recently addressed the

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importance of cooperation between hospitals in admitting patients to ICUs. They have shown that a high level of patients admission will be achieved if hospitals cooperate with each other.

Among all techniques utilized so far to study patients flow in ICUs, there is a huge lack of using differential equations and dynamical systems tools and techniques. Using such techniques gives rise to rigorous mathematical investigations on the decision-making process in ICUs. In this paper, we propose a mathematical model for the admission process in ICUs by using differential equations. We assume that all ICUs are under the control of a single organization called the Headquarters (HQ). Therefore, in our model, ICUs are connected to each other through the HQ and can cooperate in admitting patients. Fig. 1 illustrates the architecture of the model considered in this paper. According to Fig. 1, patients first enter the Emergency Departments (EDs) of hospitals. In this stage, they are either admitted to the ICU of the hospital where they first entered or referred to the HQ to be later hospitalized. If there is any hospital that is capable of giving treatments, patients are referred to the allocation list of the HQ; however, if the HQ does not manage to find a hospital instantaneously, patients are registered in the waiting list. Using differential equations, the process shown in Fig. 1 is modeled as follows

$$\begin{cases} \frac{dE}{dt'} = p' - r_1 E - cEF_{r'}(A) - EG_{r'}(A), \\ \frac{dA}{dt'} = -r_2 A + cEF_{r'}(A) + cWF_{r'}(A), \\ \frac{dW}{dt'} = -cWF_{r'}(A) + EG_{r'}(A), \end{cases} \tag{1}$$

where p' is the number of patients per unit of time entering the EDs for receiving the critical care, E represents the number of patients in the EDs, A determines the number of patients in the allocation list of the HQ, W denotes the number of patients in the waiting list of the HQ, r_1 and r_2 are the rates of transfer, c is the scaling factor, and $F_{r'}(A)$ and $G_{r'}(A)$ are bounded nonlinear functions. The scaling factor c can be viewed as the number of physicians who visit a single patient at a given time. One of the most important ingredients of a dynamical model is its nonlinear part since the emergence of many complex dynamical behaviors is dependent on nonlinear terms [7,22,29]. We represent the nonlinearity of system (1) with the two following Hill functions [21]

$$F_{r'}(A) = \frac{r'}{\alpha + A^n}, \tag{2}$$

$$G_{r'}(A) = \frac{r'A^n}{\alpha + A^n}, \tag{3}$$

where $n > 1$ is the Hill coefficient. Fig. 2 depicts the graphs of $F_{r'}(A)$ and $G_{r'}(A)$ for different values of α and selected values of the parameters n and r'. The parameters n and α define a degree of sensitivity in our model with respect to the variable A. In fact, this sensitivity sets a threshold for the effect of the variable A on the dynamics of the model and as a result, gives a regulatory role to this variable. According to Fig. 2, for the fixed value of $n = 2$, as the value of α gets smaller, the nonlinear functions (2) and (3) become steeper and as a result, the model senses more sensitivity to variations in the value of A. Since later in Section 2 the value of n will be fixed, we regard only α as the sensitivity parameter in this paper. For convenience, it is supposed that patients are transferred between different stages of the admission process with the same constant rate r'. The first term in the first equation of system (1) shows a constant increase in the number of patients in the EDs of hospitals, while the second term indicates that the number of patients in the EDs decreases due to their transfer to ICU beds. The third and fourth terms show further decrease in the number of patients because of their transfer to either the allocation or the waiting list of the HQ. The first term in the second equation of system (1) shows a decrease in the number of patients in the allocation list of the HQ as patients are admitted to a new hospital. The second and third terms though respectively indicate an increase due to the transfer of patients from the EDs and the waiting list to the allocation list. In the third equation, the first term shows that the number of patients in the waiting list diminishes as new hospitals with empty ICU beds are found, and finally according to the second term, this number goes higher as more patients in the EDs are referred to the waiting list of the HQ. In our model, it is assumed that the waiting list has an infinite capacity and patients can physically stay in

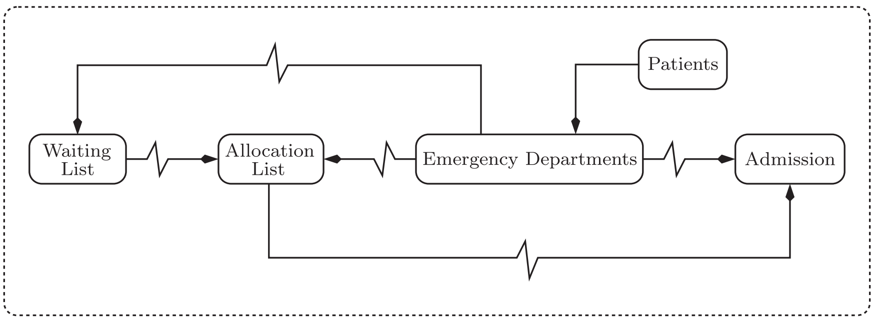

Fig. 1. Diagram of the admission process in ICUs. This diagram illustrates how the HQ monitors the entire process.