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Wavelets method for solving systems of nonlinear singular fractional Volterra integro-differential equations

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ABSTRACT

This paper presents a computational method for solving a class of system of nonlinear singular fractional Volterra integro-differential equations. First, existences of a unique solution for under studying problem is proved. Then, shifted Chebyshev polynomials and their properties are employed to derive a general procedure for forming the operational matrix of fractional derivative for Chebyshev wavelets. The application of this operational matrix for solving mentioned problem is explained. In the next step, the error analysis of the proposed method is investigated. Finally, some examples are included for demonstrating the efficiency of the proposed method.

1. Introduction

Fractional calculus has been used to model physical and engineering processes that are found to be best described by fractional differential equations. Recently, fractional calculus has attracted much attention since it plays an important role in many fields of science and engineering, for example see [1–9]. As we know, many mathematical models of real phenomenons arising in various fields of science and engineering are linear or nonlinear systems. Nevertheless, most differential systems used to describe physical phenomena are integer-order systems [10]. It is worth mentioning that with the development of fractional calculus, the behavior of many systems can be described by using the fractional differential and fractional integro-differential systems (see [11] and references therein). In this paper we consider the following system of nonlinear singular fractional Volterra integro-differential equations:

$$\begin{aligned} &D_*^{\alpha_j} y_j(t) = f_j(t, y_1(t), y_2(t), \ldots, y_n(t)) + \int_a^t (t-s)^{-\beta_j} K_j(t, s, y_1(s), y_2(s), \ldots, y_n(s)) ds \\ &y_j(a) = y_{0j}, \quad j = 1, 2, \ldots, n, \end{aligned} \tag{1}$$

where $y_j : [a,b] \to \mathbb{R}$ is the unknown function, $y_{0j} \in \mathbb{R}, f_j : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ is continues function and $K_j : [a,b] \times [a,b] \times \mathbb{R}^n \to \mathbb{R}$ also are continues functions that satisfy adequate Lipschitzian conditions, $D_*^{\alpha_j}$ is the derivative of y_j of order α_j in the Caputo sense, $\alpha_j > 0, 0 < \beta_j < 1$.

An usual way for solving functional equations is to express the solution as a linear combination of the so-called basis functions. These basis functions can for instance be orthogonal or not orthogonal bases. Approximation by orthogonal family

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of basis functions has found wide application in science and engineering [12]. The most frequently used orthogonal function are sine–cosine functions, block pulse functions, Legendre, Chebyshev and Laguerre polynomials. The main idea of using an orthogonal basis is that the problem under consideration reduces to a system of linear or nonlinear algebraic equations [12]. This can be done by truncated series of orthogonal basis function for the solution of the problem and using the operational matrices [13]. It is well known that we can approximate any analytic function, $C^\infty[a,b]$, by the eigenfunctions of certain singular Sturm–Liouville problems such as Legendre or Chebyshev orthogonal polynomials. In this manner, the truncation error approaches zero faster than any negative power of the number of basic functions used in the approximation [14]. This phenomenon is usually referred to as "spectral accuracy" [14].

Wavelets theory is a relatively new and an emerging area in mathematical research (for example see [15–18] and references therein). It has been applied in a wide range of engineering disciplines. Wavelets are localized functions, which are the basis for energy-bounded functions and in particular for $L^2(\mathbb{R})$. So that localized pulse problems can be easily approached and analyzed [15–18]. They are used in system analysis, optimal control, numerical analysis, signal analysis for wave form representation and segmentations, time–frequency analysis and fast algorithms for easy implementation [19]. However wavelets are just another basis set which offers considerable advantages over alternative basis sets and allows us to attack problems not accessible with conventional numerical methods. Their main advantages are as [20]: the basis set can be improved in a systematic way, different resolutions can be used in different regions of space, the coupling between different resolution levels is easy, there are few topological constraints for increased resolution regions, the Laplace operator is diagonally dominant in an appropriate wavelet basis, the matrix elements of the Laplace operator are very easy to calculate and the numerical effort scales linearly with respect to the system size. Chebyshev wavelets method as a specific kind of wavelets methods has been widely applied for solving functional equations, for example see [21–25]. In this communication, it is worth mentioning that Chebyshev wavelets has mutually spectral accuracy, orthogonality and other useful mentioned properties of wavelets.

The main purpose of this paper is to apply Chebyshev wavelets method for solving nonlinear systems of singular fractional Volterra integro-differential equations of type (1). First, existence of a unique solution for under study problem is proved and then some useful theorem about fractional operational matrix of derivative for Chebysheve polynomails and Chebyshev wavelets are proved. After that the operational matrix of fractional derivative of Chebysheve wavelets is applied to obtain approximate solution of (1). Error analysis of the proposed method is investigated. This paper is organized as follows: In Section 2 some necessary definitions of the fractional calculus are introduced. In Section 3 existence of a unique solution for problem under study is proved. In Section 4 Chebyshev polynomials and Chebyshev wavelets are introduced. In Section 5 the proposed method is described. In Section 6 error analysis of the proposed method is investigated. In Section 7 some numerical examples are presented. Finally a conclusion is drawn in Section 8.

2. Basic definitions

With the development of theories of fractional derivatives and integrals, many definitions appear, such as Riemann–Liouville and Caputo fractional differential–integral definition [26], which can be described as follows:

(1) Riemann–Liouville definition:

$$D^\alpha f(t)=\begin{cases}\frac{d^nf(t)}{dt^n}, & \alpha=n\in\mathbb{N},\\ \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t\frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau, & 0\leqslant n-1<\alpha<n.\end{cases}\tag{2}$$

Fractional integral of order α is defined as:

$$I^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)d\tau,\quad I^0f(t)=f(t).\tag{3}$$

(2) Caputo definition:

$$D_*^\alpha f(t)=\begin{cases}\frac{d^nf(t)}{dt^n}, & \alpha=n\in\mathbb{N},\\ \frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau, t>0, & 0\leqslant n-1<\alpha<n.\end{cases}\tag{4}$$

The useful relation between the Riemann–Liouvill operator and Caputo operator is given by the following expression:

$$I^\alpha D_*^\alpha f(t)=f(t)-\sum_{k=0}^{n-1}f^{(k)}(0^+)\frac{t^k}{k!},t>0,(n-1<\alpha\leqslant n).\tag{5}$$

Also for the Caputo's derivative we have $D_*^\alpha c=0$, in which c is a constant and

$$D_*^\alpha t^n=\begin{cases}0, & n<\lceil\alpha\rceil,n\in\mathbb{Z}^+\\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}t^{n-\alpha}, & n\geqslant\lceil\alpha\rceil,n\in\mathbb{Z}^+\end{cases},\tag{6}$$

where we use the ceiling function $\lceil\alpha\rceil$ to denote the smallest integer greater than or equal to α. For more details about fractional calculus and its properties see [26].