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On localized mixing in action–action–angle flows

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ABSTRACT

Tailored mixing inside individual droplets could be useful to ensure that reactions within microscopic discrete fluid volumes, which are used as microreactors in "digital microfluidic" applications, take place in a controlled fashion. We consider a translating spherical liquid drop to which we impose a time periodic rigid-body rotation. Such a rotation induces mixing via chaotic advection in a narrow area inside the drop. We show that chaotic advection is caused by the resonance destruction of adiabatic invariants and derive an analytical description for the size of the mixing region.

1. Introduction

Efficient and controlled mixing in 3D Stokes flows, that is so important in many microfluidic settings, is difficult to achieve because the flows are laminar. Among different flow settings, microfluidics in droplets holds great promise for various applications and was studied extensively experimentally and numerically, [2,9,10,20,21,14,8,6,11], and also analytically as discussed below.

In many settings, mixing must be achieved within the confinement of near-integrable flows. Typically the base (or unperturbed) flow is autonomous and possesses a certain degree of symmetry, having one or two invariants. Properties of the flows with one invariant, so-called action–angle–angle flows, are well described by the KAM theory (see, e.g., [1]). Alternatively, the base flow may possess two invariants. Such flows are called action–action–angle flows, and are quite common as any axisymmetric Stokes flow belongs to this class: the azimuthal angle and the streamfunction are the invariants. The first study of mixing in such flows under time-periodic perturbations is due to Feingold and co-workers, [7,13], who recognized that mixing is caused by the breakdown of adiabatic invariance near surfaces where the frequency of the perturbation is in resonance with the frequency of the base flow. Since then mixing in near-integrable flows was studied quite widely. It is typical that the presence of singular surfaces (like resonances or separatrices) leads to mixing over a large (of the order of the whole system) chaotic mixing region (CMR) (see, e.g., [2,15,18,19]). Contrarily to that, Kroujiline and Stone, [9], introduced a perturbed action–action–angle flow inside a drop where mixing was localized near the axis and near the surface of the drop. In our previous publication, [3], we introduced an action–action–angle flow that had a CMR localized near any resonance tori and thus can be placed anywhere in the flow. The key property of such a flow is that the frequency of the unperturbed flow is a function of one of the invariants (actions) of the unperturbed flow *and* this very invariant is an adiabatic invariant of the perturbed flow. However, in [3] and subsequent publications, the size of the CMR was computed only numerically. In the present paper we provide an analytical estimation of the size of the CMR for small values of the perturbation.

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2. Flow structure

In the present paper we consider a flow introduced in [3] and similar to the one considered in [9]: a spherical Newtonian drop immersed in an incompressible Newtonian flow. We assume that the local $Re \ll 1$ and that the interfacial tension is sufficiently large for the drop to remain spherical. The time-dependence is introduced by considering the time-periodic external vorticity. In a Cartesian coordinate system translating with the center-of-mass velocity of the drop, and with the z-axis in the direction of the translation, the motion of passive tracers inside the droplet are described by a non-autonomous dynamical system, [3]:

$$\begin{aligned} \dot{x} &= zx - a(t)\omega_z y, \\ \dot{y} &= zy + a(t)(\omega_z x - \omega_x z), \\ \dot{z} &= (1 - 2x^2 - 2y^2 - z^2) + a(t)\omega_x y. \end{aligned} \tag{1}$$

In (1), all the lengths and velocities have been non-dimensionalized by the drop radius and the magnitude of the translational velocity. The external vorticity is defined by $(\omega_x, \omega_y, \omega_z) = (1/\sqrt{2}, 0, 1/\sqrt{2})$, the unitary vector defines the axis of rotation. The magnitude of the perturbation,

$$a(t) = \varepsilon/2(1 + \cos \omega t), \tag{2}$$

is characterized by the frequency ω and the amplitude ε. In this paper, we consider only small amplitudes ($\varepsilon \ll 1$). The case of large ε was considered numerically in the previous publications, [3,3,5]. Flow (1) is the superposition of a Hill's vortex and an unsteady rigid body rotation. The surface of the drop, $r^2 = x^2 + y^2 + z^2 = 1$, is invariant under flow (1).

The case $\varepsilon = 0$ represents the unperturbed (or base) flow, which is axisymmetric and possesses two invariants, the streamfunction ψ and the azimuthal angle ϕ:

$$\psi = \rho^2(1 - r^2)/2, \quad \tan \phi = y/x,$$

where $\rho^2 = x^2 + y^2$. The streamfunction ψ is equal to zero on the surface of the sphere and on the z-axis, and has the maximum value of 1/8 along the circle of degenerate fixed points ($\rho = 1/\sqrt{2}, z = 0$). A schematic view of the unperturbed flow in shown in Fig. 1. Almost all the streamlines inside the drop are closed curves, residing in the vertical planes $\phi = \text{const}$, and are symmetric with respect to the equatorial plane. The frequency Ω of periodic motion along a given streamline is defined as

$$\frac{2\pi}{\Omega(\psi)} = \int_{-\pi/2}^{\pi/2} \frac{\sqrt{2}}{\sqrt{1 + \sqrt{1 - 8\psi} \sin \alpha}} \, d\alpha. \tag{3}$$

To specify a point on an unperturbed streamline, we introduce a uniform phase χ mod(2π), such that $\chi = 0$ at the point on the equatorial plane that is closer to the vertical axis. The word "uniform" means that χ changes with a uniform (constant) rate everywhere on a given streamline: $\dot{\chi} = \Omega(\psi)$. Note that there are only two locations where χ can be immediately analytically obtained: $\chi = 0$ and $\chi = \pi$, when the streamline crosses the (x, y) plane. Fig. 1.

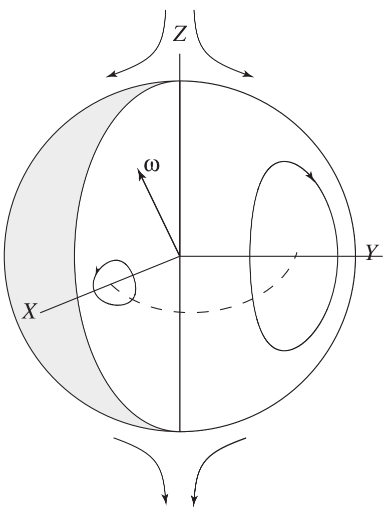

Fig. 1. Schematic view of the unperturbed flow. Vector ω denotes the external vorticity. Two characteristic streamlines are shown. The dashed line is the circle of degenerate elliptic fixed points.