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A new discrete mechanics approach to swing-up control of the cart-pendulum system

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ABSTRACT

This paper develops a new swing-up control method for the cart-pendulum system via discrete mechanics. The swing-up control law consists of two parts: the swing-up stage and the stabilization one. In the swing-up stage, we use a controller based on a discrete Lyapunov function and it can swing up the pendulum. Then, in the stabilization stage, we utilize a stabilizing controller based on the linearized system and discrete-time optimal regulator theory. In addition, transformation methods from discrete control inputs into continuous zero-order hold inputs are introduced. From some simulation results, we can confirm that the cart-pendulum system is swung up and stabilized by our new method.

1. Introduction

Recently, *discrete mechanics* has been focused on as a new discretizing tool and a new numerical solution method [1–5]. In discrete mechanics, some important fundamental concepts of classical mechanics are discretized, and discrete Hamilton's principle, discrete Lagrange–d'Alembert principle and discrete Euler–Lagrange equations are introduced. It is known that discrete mechanics has some interesting characteristics (see [3,4] for the details); (i) discrete mechanics shows less numerical errors (approximation error) in comparison with other numerical solutions of a same order such as the Euler method and the Runge–Kutta method, (ii) it can describe energies for both conservative and dissipative systems with less errors, (iii) the discrete model (discrete Euler–Lagrange equation) has a property of the symplectic mapping and the implicit numerical solution, (iv) some laws of physics such as Noether's theorem are satisfied, (v) numerical simulations for larger sampling times indicate can work and show good calculation results with less errors, Hence, we can say that discrete mechanics has a possibility of controller synthesis with high compatibility with computers, and researches on theoretical analysis and applications of discrete mechanics are now in progress.

In the past, some studies on applications of discrete mechanics have been done; controller synthesis from the standpoint of controlled Lagrangian [6,7], optimal control based on discrete mechanics and an application to hovercrafts [5]. We have derived some results on discrete mechanics from both analysis and synthesis aspects as follows: solvability analysis of implicit discrete Euler–Lagrange equations and stabilization control for the discrete cart-pendulum [11,18], transformation to continuous control inputs and stabilization control for the continuous cart-pendulum system [12,19], experimental evaluation of discrete mechanics for the actual cart-pendulum system [13,19], applications to gait generation for the compass-type

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biped robot [14–17]. In these studies, we have shown the effectiveness of discrete mechanics for control theory and possibility of applications for mechanical systems. However, some of these work deal with only a stabilization problem of the cart-pendulum system, which is included in the field of linear control problems. If we demonstrate the effectiveness of discrete mechanics for nonlinear control problems, we have to treat a swing-up control problem for the cart-pendulum system, which is one of the difficult benchmark problems in nonlinear control theory [24–27,29,30]. For the swing-up control problem of the cart-pendulum system and the pendubot, some control methods based on the models derived by normal continuous mechanics have been proposed; the energy control method [8], a control strategy based on an energy of the system [9] and so on. However, A swing-up control strategy based on discrete mechanics had not been proposed. Recently, [10] proposes a swing-up control method based on *discrete mechanics and optimal control* [5] for the double pendulum system.

Hence, the main purpose of this paper is to develop a new swing-up control method for the cart-pendulum system from the viewpoint of discrete mechanics, which is totally different from the method in [10]. Investigation on the application potentiality of discrete mechanics to nonlinear control theory is also another important purpose of this paper. This paper is organized as follows. In Section 2, some fundamental concepts on discrete mechanics are summarized. We also discuss a condition that the linearly approximated systems of the discrete Euler–Lagrange systems is represented as explicit equations. Next, Section 3 derives mathematical models of both continuous and discrete cart-pendulum systems via continuous and discrete mechanics, respectively. In Section 4, we then develop a new swing-up control method based on discrete mechanics for the discrete cart-pendulum. The swing-up controller consists of 2 stages: the swing-up stage and the stabilization one. We also discuss a transformation method of a discrete control input into a zero-order hold input, which can control the continuous cart-pendulum system. Finally, in order to confirm the effectiveness of our new method, some numerical simulations are illustrated in Section 5.

2. Discrete mechanics

2.1. Basic concepts of discrete mechanics

This section summarizes some basic concepts in discrete mechanics. For more details about discrete mechanics, see [1–5].

Let Q be an n-dimensional configuration manifold and $q \in \mathbf{R}^n$ be a generalized coordinate of Q. We also refer to T_qQ as the tangent space of Q at a point $q \in Q$ and $\dot{q} \in T_qQ$ denotes a generalized velocity. Moreover, we consider a time-invariant Lagrangian as $L^c(q, \dot{q}) : TQ \to \mathbf{R}$. We first explain about the discretization method. The time variable $t \in \mathbf{R}$ is discretized as $t = kh$ $(k = 0, 1, 2, \ldots)$ by using a sampling interval $h > 0$. We denote q_k as a point of Q at the time step k, that is, a curve on Q in the continuous setting is represented as a sequence of points $q^d := \{q_k\}_{k=0}^N$ in the discrete setting. The transformation method of discrete mechanics is carried out by the replacement:

$$q \approx (1-\alpha)q_k + \alpha q_{k+1}, \quad \dot{q} \approx \frac{q_{k+1} - q_k}{h}, \tag{1}$$

where q is expressed as a internally dividing point of q_k and q_{k+1} with an internal division ratio α $(0 < \alpha < 1)$. We then define *a discrete Lagrangian*:

$$L_\alpha^d(q_k, q_{k+1}) := hL\Big((1-\alpha)q_k + \alpha q_{k+1}, \frac{q_{k+1} - q_k}{h}\Big) \tag{2}$$

and *a discrete action sum*:

$$S_\alpha^d(q_0, q_1, \cdots, q_N) = \sum_{k=0}^{N-1} L_\alpha^d(q_k, q_{k+1}). \tag{3}$$

We next summarize the discrete equations of motion. Consider a variation of points on Q as $\delta q_k \in T_{q_k}Q$ $(k = 0, 1, \ldots, N)$ with the fixed condition $\delta q_0 = \delta q_N = 0$. In analogy with the continuous setting, we define a variation of the discrete action sum (3) as

$$\delta S_\alpha^d(q_0, q_1, \cdots, q_N) = \sum_{k=0}^{N-1} \delta L_\alpha^d(q_k, q_{k+1}) \tag{4}$$

as shown in Fig. 1. The discrete Hamilton's principle states that *only a motion which makes the discrete action sum (3) stationary is realized*. Calculating (4), we have

$$\delta S_\alpha^d = \sum_{k=1}^{N-1} \{D_1 L_\alpha^d(q_k, q_{k+1})\delta q_k + D_2 L_\alpha^d(q_{k-1}, q_k)\}\delta q_k, \tag{5}$$

where D_1 and D_2 denotes the partial differential operators with respect to the first argument and the second one, respectively. Consequently, from the discrete Hamilton's principle and (5), we obtain *the discrete Euler–Lagrange equations*:

$$D_1 L_\alpha^d(q_k, q_{k+1}) + D_2 L_\alpha^d(q_{k-1}, q_k) = 0, \quad k = 1, \ldots, N-1, \tag{6}$$