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A dynamic system model for solving convex nonlinear optimization problems

A. R. Nazemi *

Department of Mathematics, School of Mathematical Sciences, Shahrood University of Technology, P.O. Box 3619995161-316, Shahrood, Iran

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ABSTRACT

This paper proposes a feedback neural network model for solving convex nonlinear programming (CNLP) problems. Under the condition that the objective function is convex and all constraint functions are strictly convex or that the objective function is strictly convex and the constraint function is convex, the proposed neural network is proved to be stable in the sense of Lyapunov and globally convergent to an exact optimal solution of the original problem. The validity and transient behavior of the neural network are demonstrated by using some examples.

1. Introduction

The convex programming problems arise often in mathematics, engineering and finance problems [2]. Many publications deal with this problem by numerical algorithms, see for instance [1–5] and references therein. Since the computing time greatly depends on the dimension and the structure of the problems, numerical algorithm is usually less effective in large-scale or real-time optimization problems. The artificial neural networks (ANN), on the other hand, have massively paralleled distributed computation and fast convergence. It can be considered as an efficient method to solve large-scale or real-time optimization problems.

The essence of neural network approach for optimization is to establish an energy function (nonnegative) and a dynamic system which is a representation of an artificial neural network [6]. The dynamic system is normally in the form of first order ordinary differential equations. It is expected that for an initial point, the dynamic system will approach its static state (or equilibrium point) which corresponds the solution of the underlying optimization problem. An important requirement is that the energy function decreases monotonically as the dynamic system approaches an equilibrium point. Recently, neural networks for solving optimization problem have been rather extensively studied and some important results have also been obtained in [7–25].

Motivated by the above discussions, in this paper, we present the optimization techniques for solving problem (1) based on the neural network method. Based on the Karush Kuhn Tucker (KKT) optimality conditions of convex programming, a neural network model for solving the CNLP will be proposed. The existence of solutions of the neural network is discussed. This neural network is proved to be globally stable by constructing a suitable Lyapunov function and the solution trajectory can converge to an optimal solution of the original optimization problem.

The paper is organized as follows. In Section 2, we will construct corresponding neural network model. In Section 3, we will investigate the stability of the equilibrium point and the convergence of the optimal solution. In Section 4, an illustrative

* Tel./fax: +98 273 3392012.
E-mail address: nazemi20042003@yahoo.com

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several examples and simulation results will be given to show the effectiveness of the proposed method. Conclusions are given in Section 5.

2. A dynamic system model

Let the following be a general CNLP problem:

$$\text{minimize} \quad f(x) \tag{1}$$
$$\text{subject to} \quad g(x) \leqslant 0, \tag{2}$$
$$h(x) = 0, \tag{3}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$, $g(x) = (g_1(x), g_2(x), \ldots, g_m(x))^T$ is an m-dimensional vector-valued continuous function of n variables, and the functions f, $g_1,\ldots,g_m$ are assumed to be convex and twice differentiable, $h(x) = Ax - b$, $A \in \mathbb{R}^{l\times n}$, $\operatorname{rank}(A) = l(0 \leqslant l < n)$ and $b \in \mathbb{R}^l$. Throughout this paper, we assume that (1)–(3) has unique optimal solution.

For the convenience of later discussions, it is necessary to introduce to a few notations, definitions and two lemmas. In what follows, $\|\cdot\|$ denotes l^2-norm of $\mathbb{R}^n$ and $x = (x_1,x_2,\ldots,x_n)^T$. If a function $f : \mathbb{R}^n \to \mathbb{R}$, then $\nabla f \in \mathbb{R}^n$ stands for its gradient.

Definition 2.1. A function $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^n$ is said to be Lipschitz continuous with constant L on a set $\mathbb{R}^n$ if, for each pair of points $x, y \in \mathbb{R}^n$,

$$\|\mathcal{F}(x) - \mathcal{F}(y)\| \leqslant L\|x - y\|.$$

$\mathcal{F}$ is said to be locally Lipschitz continuous on $\mathbb{R}^n$ if each point of $\mathbb{R}^n$ has a neighborhood $D_0 \subset \mathbb{R}^n$ such that the above inequality holds for each pair of points x, $y \in D_0$.

Definition 2.2. A mapping $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^n$ is monotone if

$$(x - y)^T(\mathcal{F}(x) - \mathcal{F}(y)) \geqslant 0, \quad \forall x, y \in \mathbb{R}^n.$$

$\mathcal{F}$ is said to be strictly monotone, if the strict inequality holds whenever $x \neq y$.

Lemma 2.3 [26]. *If a mapping $\mathcal{F}$ is continuously differentiable on an open convex set D including Ω, then $\mathcal{F}$ is monotone (strictly monotone) on Ω, if and only if its Jacobian matrix $\nabla\mathcal{F}(x)$ is positive semidefinite (positive definite) for all $x \in \Omega$.*

Definition 2.4. Let $x(t)$ be a solution trajectory of a system $x' = \mathcal{F}(x)$, and let X^* denotes the set of equilibrium points of this equation. The solution trajectory of the system is said to be globally convergent to the set X^*, if $x(t)$ satisfies

$$\lim_{t\to\infty} \operatorname{dist}(x(t), X^*) = 0,$$

where $\operatorname{dist}(x(t), X^*) = \inf_{y\in X^*}\|x - y\|$. In particular, if the set X^* has only one point x^*, then $\lim_{t\to\infty}x(t) = x^*$, and the system is said to be globally asymptotically stable at x^* if the system is also stable at x^* in the sense of Lyapunov.

Consider the Lagrange function of (1)–(3) similar to [15] as follows:

$$L(x, u, v) = f(x) + \frac{1}{2}\sum_{k=1}^{m} u_k^2 g_k(x) + \sum_{p=1}^{l} v_p h_p(x), \tag{4}$$

where $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^l$ are called as the Lagrange multipliers. It is well known [2] that $x^* \in \mathbb{R}^n$ is an optimal solution of (1)–(3) if and only if there exist $u^* \in \mathbb{R}^m$ and $v^* \in \mathbb{R}^l$ such that $\left(x^{*T}, u^{*T}, v^{*T}\right)^T$ satisfies the following KKT system

$$\begin{cases} u^* \geqslant 0, \quad g(x^*) \leqslant 0, \quad u^{*T}g(x^*) = 0, \\ \nabla f(x^*) + \nabla g(x^*)^T u^* + \nabla h(x^*)^T v^* = 0, \\ h(x^*) = 0. \end{cases} \tag{5}$$

x^* is called a KKT point of (1)–(3) and a pair $\left(u^{*T}, v^{*T}\right)^T$ is called the Lagrangian multiplier vector corresponding to x^*.

Lemma 2.5 [2]. *If f and g_k, $k = 1,\ldots,m$ are all convex, then x^* is an optimal solution of (1)–(3) if and only if x^* is a KKT point of (1)–(3).*

Now, let $x(\cdot)$, $u(\cdot)$ and $v(\cdot)$, to be some time dependent variables. The aim is to construct a continuous-time dynamical system that will settle down to the KKT point of nonlinear programming problem (1)–(3). Therefore, our aim now is to design a neural network that will settle down to the saddle point of $L(x,u,v)$. We may describe the neural network model corresponding to (1)–(3) and its dual by the following nonlinear dynamical system