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Existence of solutions for fractional differential equations with multi-point boundary conditions ☆

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ABSTRACT

We discuss the existence of solutions for a nonlinear multi-point boundary value problem of integro-differential equations of fractional order $q \in (1,2]$. Our analysis relies on the contraction mapping principle and the Krasnoselskii's fixed point theorem. Example is provided to illustrate the theory.

1. Introduction

This paper is mainly concerned with the existence and uniqueness results for the following fractional differential equation with multi-point boundary conditions

$$\begin{cases} {}^cD^q_{0+}u(t) = f(t, u(t), (Ku)(t), (Hu)(t)), \quad t \in (0,1), \\ a_1u(0) - b_1u'(0) = d_1u(\xi_1), a_2u(1) + b_2u'(1) = d_2u(\xi_2), \end{cases} \tag{1}$$

where $1 < q \leqslant 2$ is a real number, $J = [0,1]$, ${}^cD^q_{0+}$ is the Caputo's fractional derivative, and $a_1, b_1, d_1, a_2, b_2, d_2$ are real numbers, $0 < \xi_1,\ \xi_2 < 1$, $f : J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function satisfying some assumptions that will be specified later. By $\mathbb{C} = C(J, \mathbb{R})$ denote the Banach space of all continuous mapping $u : J \to \mathbb{R}$ with norm $\|u\| := \sup\{|u(t)| : t \in J\}$, and K, H: $J \times J \to [0, \infty)$,

$$(Ku)(t) = \int_0^t k(t,s)u(s)ds, \quad (Hu)(t) = \int_0^t h(t,s)u(s)ds$$

with $k_0 = \sup_{t\in J}|\int_0^t k(t,s)ds|,\ h_0 = \sup_{t\in J}|\int_0^t h(t,s)ds|$.

Recently, fractional differential equations have found numerous applications in various fields of physics and engineering [1,2]. It should be noted that most of the books and papers on fractional calculus are devoted to the solvability of initial value problems for differential equations of fractional order. In contrast, the theory of boundary value problems for nonlinear fractional differential equations has received attention quite recently and many aspects of this theory need to be explored. For more details and examples, see [3–20] and the references therein. However, as far as we know, few results can be found

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in the literature concerning multi-point boundary value problems for differential equations of fractional order [21–23]. As a result, the goal of this paper is to enrich this academic area. Motivated by the above mentioned papers, the purpose of this paper is to establish the existence and uniqueness results for the boundary value problem (1) by virtue of the contraction mapping principle and the Krasnoselskii's fixed point theorem [24].

The remainder of this article is organized as follows. In Section 2, we provide some basic definitions and various lemmas which are needed later. In Section 3, we give main results of the problem (1). In Section 4, we provide an example illustrating the main result.

2. Preliminaries

In this section, we present some basic notations, definitions and preliminary results which will be used throughout this paper.

Definition 2.1 ([3,7]). The fractional integral of order $q > 0$ of a function $f : (0,\infty) \to \mathbb{R}$ is defined by

$$I^q_{0_+} f(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds$$

provided the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.2 ([3,7]). The Riemann–Liouville fractional derivative of order $q > 0$ of a continuous function $f : (0,\infty) \to \mathbb{R}$ is defined to be:

$$\left(D^q_{a+} h\right)(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-q-1} h(s)ds.$$

Here $n = [q] + 1$ and $[q]$ denotes the integer part of q.

Definition 2.3 ([3,7]). The fractional derivative of a function f in the Caputo sense is defined as

$$\left({}^cD^q_{a+} f\right)(t) = \frac{1}{\Gamma(n-q)} \int_a^t (t-s)^{n-q-1} f^{(n)}(s)ds, n-1 < q < n,$$

where $n = [q] + 1$.

Lemma 2.1 ([3,4,18]). *For $q > 0$, the general solution of the fractional differential equation ${}^cD^q_{0+}u(t) = 0$ is given by*

$$u(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R},\ i = 0,1,2,\ldots,n-1 (n = [q]+1)$.

Lemma 2.2 ([3,4,18]). *Assume that $h \in C(0,1) \cap L(0,1)$ with a derivative of order $q > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I^q_{0+}\, {}^cD^q_{0+} h(t) = h(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}$$

for some $c_i \in \mathbb{R},\ i = 0,1,2,\ldots,n-1$, where $n = [q] + 1$.

Lemma 2.3. *Let $h \in C[0,1]$ and $1 < q \leqslant 2$, then the unique solution of*

$$\begin{cases} {}^cD^q_{0+}u(t) = h(t), \quad t \in (0,1), \\ a_1 u(0) - b_1 u'(0) = d_1 u(\xi_1), a_2 u(1) + b_2 u'(1) = d_2 u(\xi_2), \end{cases} \tag{2}$$

is given by

$$\begin{aligned} u(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s)ds + g_1(t) \int_0^{\xi_1} \frac{(\xi_1 - s)^{q-1}}{\Gamma(q)} h(s)ds \\ & + g_2(t)\left[-\frac{a_2}{\Gamma(q)} \int_0^1 (1-s)^{q-1} h(s)ds - \frac{b_2}{\Gamma(q-1)} \int_0^1 (1-s)^{q-2} h(s)ds + \frac{d_2}{\Gamma(q)} \int_0^{\xi_2} (\xi_2 - s)^{q-1} h(s)ds\right]. \end{aligned} \tag{3}$$

where

$$\Delta = [(b_1 + d_1\xi_1)(a_2 - d_2) + (a_2 + b_2 - d_2\xi_2)(a_1 - d_1)] \neq 0, \tag{4}$$

$$g_1(t) = \frac{d_1}{\Delta}[a_2(1-t) + b_2 + d_2(t - \xi_2)], \quad g_2(t) = \frac{1}{\Delta}[(b_1 + d_1\xi_1) + t(a_1 - d_1)].$$