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Short communication

Exact solutions of the generalized $K(m,m)$ equations

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ABSTRACT

Family of equations, which is the generalization of the $K(m,m)$ equation, is considered. Periodic wave solutions for the family of nonlinear equations are constructed.

1. Introduction

Seeking to understand the role of nonlinear dispersion in the formation of patterns in liquid drops in 1993 Rosenau and Hyman [1] introduced a family of fully nonlinear $K(m,n)$ equations and also presented solutions of the $K(2,2)$ equation to illustrate the remarkable behavior of these equations. The $K(m,n)$ equations have the property that for certain m and n their solitary wave solutions have compact support. That is, they vanish identically outside a finite core region. These properties have a wide application in the fields of Physics and Mathematics, such as Nonlinear Optics, Geophysics, Fluid Dynamics and others. Later, this equation was studied by various scientists worldwide [2–14].

In this paper we construct periodic wave solutions for the following family of nonlinear partial differential equations:

$$\frac{\partial u}{\partial t}+\sum_{k=0}^{N}\alpha_k\frac{\partial^{2k+1}u^m}{\partial x^{2k+1}}=0,\quad N\geqslant 1,\ m\neq 1,\ \alpha_k\neq 0. \tag{1.1}$$

Eq. (1.1) is of order $2N+1$ and depends on $N+2$ parameters denoted by $\alpha_0,\ldots,\alpha_N,m$. This family contains a number of well-known generalizations of partial differential equations which were considered before [15–29].

This paper is organized as follows. In Section 2 we describe a method which enables one to construct periodic wave solutions for the concerned family of nonlinear partial differential equations. In Sections 3–6 we give several specific examples for some meanings of N.

2. Method applied

Applying traveling wave variable:

$$u(x,t)=y(z),\quad z=x-C_0t, \tag{2.1}$$

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to Eq. (1.1) and integrating the results yield the following Nth-order equation:

$$\sum_{k=0}^{N}\alpha_k\frac{d^{2k}y^m}{dz^{2k}}-C_0y=0,\quad N\geqslant 1,\ m\neq 1,\ \alpha_k\neq 0. \tag{2.2}$$

The constant of integration is set to be zero. Substituting $y(z)=F(z)^p$ into:

$$\alpha_n\frac{d^{2N}y^m}{dz^{2N}}-C_0y=0,$$

we have $p=\frac{2N}{m-1}$. Note that Eq. (2.2) is an autonomous equation, and we can substitute z to $(z-z_0)$. We will take this fact into account in final solution, but we omit this substitution in our calculations. We search solutions of Eq. (2.2) in the form:

$$y(z)=(A_N)^{\frac{1}{m-1}}\cos^{\frac{2N}{m-1}}(B_N(z-z_0)). \tag{2.3}$$

There is a remarkable property of a function $\cos(B_Nz)$. First of all we have to show expansion terms of Eq. (2.2).

In the case $k=1$ we have the following expression:

$$\frac{d^2}{dz^2}\cos^{\frac{2m}{m-1}}(B_1z)=-\frac{(2mB_1)^2}{(m-1)^2}\cos^{\frac{2m}{m-1}}(B_1z)+\frac{2m(m+1)B_1^2}{(m-1)^2}\cos^{\frac{2}{m-1}}(B_1z). \tag{2.4}$$

In the case $k=2$ we obtain:

$$\frac{d^4}{dz^4}\cos^{\frac{4m}{m-1}}(B_2z)=\frac{(4mB_2)^4}{(m-1)^4}\cos^{\frac{4m}{m-1}}(B_2z)-\frac{16mB_2^4(15m^3+11m^2+5m+1)}{(m-1)^4}\cos^{\frac{2(m+1)}{m-1}}(B_2z)$$
$$+\frac{8mB_2^4(3m+1)(m+3)(m+1)}{(m-1)^4}\cos^{\frac{4}{m-1}}(B_2z). \tag{2.5}$$

In the case $k=3$ we get:

$$\frac{d^6}{dz^6}\cos^{\frac{4m}{m-1}}(B_3z)=-\frac{(6mB_3)^6}{(m-1)^6}\cos^{\frac{6m}{m-1}}(B_3z)+\frac{96mB_3^6(5m+1)(7m^2+m+1)(19m^2+7m+1)}{(m-1)^6}\cos^{\frac{2(2m+1)}{m-1}}(B_3z)$$
$$-\frac{144mB_3^6(2m+1)(5m+1)(m+1)(14m^2+8m+5)}{(m-1)^6}\cos^{\frac{2(m+2)}{m-1}}(B_3z)$$
$$+\frac{72mB_3^6(5m+1)(m+1)(m+5)(m+2)(m+1)}{(m-1)^6}\cos^{\frac{6}{m-1}}(B_3z). \tag{2.6}$$

In the general case $k=N$ derivative takes the form:

$$\frac{d^{2N}}{dz^{2N}}\cos^{\frac{2Nm}{m-1}}(B_Nz)=(-1)^N\frac{(2NmB_N)^{2N}}{(m-1)^{2N}}\cos^{\frac{2Nm}{m-1}}(B_Nz)+(-1)^{N+1}\frac{B_N^{2N}M_1^{2N}}{(m-1)^{2N}}\cos^{\frac{2(Nm-m+1)}{m-1}}(B_Nz)$$
$$+(-1)^N\frac{B_N^{2N}M_2^{2N}}{(m-1)^{2N}}\cos^{\frac{2(Nm-2m+2)}{m-1}}(B_Nz)+(-1)^{N+1}\frac{B_N^{2N}M_3^{2N}}{(m-1)^{2N}}\cos^{\frac{2(Nm-3m+3)}{m-1}}(B_Nz)$$
$$+\cdots-\frac{B_N^{2N}M_{N-1}^{2N}}{(m-1)^{2N}}\cos^{\frac{2(Nm-(N-1)m+N-1)}{m-1}}(B_Nz)+\frac{B_N^{2N}M_N^{2N}}{(m-1)^{2N}}\cos^{\frac{2N}{m-1}}(B_Nz), \tag{2.7}$$

where $M_1^{2N},\ldots,M_N^{2N}$ are polynomials of $2N$ power.

Substituting Eq. (2.7) into Eq. (2.2) we obtain the expression:

$$A_N\left(\alpha_0-\frac{(2NmB_N)^2}{(m-1)^2}\alpha_1+\frac{(2NmB_N)^4}{(m-1)^4}\alpha_2-\cdots+(-1)^N\frac{(2NmB_N)^{2N}}{(m-1)^{2N}}\alpha_N\right)\cos^{\frac{2Nm}{m-1}}(B_Nz)$$
$$+A_N\left(\frac{B_N^2M_1^2}{(m-1)^2}\alpha_1-\frac{B_N^4M_1^4}{(m-1)^4}\alpha_2+\cdots+(-1)^{N+1}\frac{B_N^{2N}M_1^{2N}}{(m-1)^{2N}}\alpha_N\right)\cos^{\frac{2(Nm-m+1)}{m-1}}(B_Nz)$$
$$+A_N\left(\frac{B_N^4M_2^2}{(m-1)^4}\alpha_2-\frac{B_N^6M_2^6}{(m-1)^6}\alpha_3+\cdots+(-1)^N\frac{B_N^{2N}M_2^{2N}}{(m-1)^{2N}}\alpha_N\right)\cos^{\frac{2(Nm-2m+2)}{m-1}}(B_Nz)+\cdots$$
$$+A_N\left(\frac{B_N^{2(N-1)}M_{N-1}^{2(N-1)}}{(m-1)^{2(N-1)}}\alpha_{N-1}-\frac{B_N^{2N}M_{N-1}^{2N}}{(m-1)^{2N}}\alpha_N\right)\cos^{\frac{2(Nm-(N-1)m+N-1)}{m-1}}(B_Nz)+\left(\frac{B_N^{2N}M_N^{2N}}{(m-1)^{2N}}\alpha_NA_N-C_0\right)\cos^{\frac{2N}{m-1}}(B_Nz)=0. \tag{2.8}$$

Equating coefficients at powers of $\cos(B_Nz)$ to zero yields an algebraic system. Solving this system we obtain the values of parameters A_N, B_N and correlations on the coefficients $\alpha_0,\ldots,\alpha_N$.