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Numerical solution of nonlinear two-dimensional integral equations using rationalized Haar functions

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ABSTRACT

Two-dimensional rationalized Haar (RH) functions are applied to the numerical solution of nonlinear second kind two-dimensional integral equations. Using bivariate collocation method and Newton–Cotes nodes, the numerical solution of these equations is reduced to solving a nonlinear system of algebraic equations. Also, some numerical examples are presented to demonstrate the efficiency and accuracy of the proposed method.

1. Introduction

Compared with the abundant literature concerned with the numerical analysis of one-dimensional integral equations (see, for example, [1–3]), the analysis of computational methods for several-dimensional integral equations has started more recently and is not so well developed, specially in the nonlinear case.

However, significant progress in this area has been obtained in the last 20 years, starting with the well-known work by Brunner and Kauthen [4], who introduced collocation and iterated collocation methods for the solution of two-dimensional linear Volterra integral equations (VIE). Kauthen has extended this study to the case of linear Volterra–Fredholm integral equations (VFIE) [5] and Brunner has considered in [6] the case of nonlinear VIE.

Important contributions to this field can be also found in the works of Guogiang Han and his co-authors. They have obtained asymptotic error expansions for different classical methods, when applied to two-dimensional integral equations, and used them as a basis to introduce extrapolation algorithms. In [7], they have used this approach to analyze the solution of linear VFIE by the Nystrom trapezoidal method. In [8,9], the iterated collocation method was applied to the solution of nonlinear VIE. Nonlinear Fredholm integral equations (FIE) have been considered in [10,11], where these equations were solved, respectively, by the Nystrom and the iterated Galerkin methods. The iterated Galerkin method was also applied in [12] to the solution of linear VIE.

Finally, we would like to cite two recent works, which will be mentioned in Section 6, where we will compare our numerical results with those obtained by different authors: Maleknejad et al. have considered the use of a basis of block-pulse functions for the numerical solution of nonlinear two-dimensional VIE [13]. In [14], Tari and Shahmorad have developed a method based on a basis of orthogonal polynomials for the numerical solution of linear two-dimensional VIE.

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Computational methods based on the application of different sets of basis functions have become a very common tool for the solution of different kinds of functional equations, including integral ones. In particular, we are interested here in the use of the so-called Haar functions (or wavelets).

The orthogonal set of Haar wavelets is a group of square waves with magnitude $-2^{j/2}$, $2^{j/2}$ and 0, $j = 0, 1, \ldots$ (for a rigorous definition, see [15]). Lynch and Reis [16] have rationalized the Haar transform by deleting the irrational numbers and introducing the integral powers of two. The modification results in what is called the rationalized Haar transform. The RH transform preserves all properties of the original Haar transform. The corresponding functions are known as RH functions. The RH functions are composed of only three amplitudes +1, −1 and 0.

An interesting property of the system of Haar functions is that the Fourier expansion of any continuous function f in this basis (Fourier-Haar series) converges to this function uniformly on [0,1] [15]. Moreover, if $\omega(\sigma,f)$ is the modulus of continuity of f and $S_n(f)$ is the n-th order partial sum of the considered series, the following inequality holds true:

$$\sup_{0\leqslant t\leqslant 1}|f(t)-S_n(t)| \leqslant 12\omega\left(\frac{1}{n},f\right), \quad n=1,2,\ldots.$$

An interesting feature of the Haar functions, which makes them attractive from the computational point of view, is that the computation of integrals of such functions is very easy. Therefore, when solving numerically differential or integral equations, the use of a basis of Haar functions is very advantageous, when compared with other basis of orthogonal functions. On the other hand, the convergence rate of such methods can be rather competitive.

This explains why many authors have preferred wavelet methods when solving different kinds of functional equations. For example, Okhita and Kobayashi have applied Haar wavelets to the solution of ordinary differential equations [17,18]. In [19], the same authors have utilized rationalized Haar functions to approximate the solutions of partial differential equations and they have pointed out the advantage of using such basis functions for the representation of two-variable functions.

The numerical solution of nonlinear one-dimensional Fredholm equations using a basis of Haar functions was considered by Razzaghi and Ordokhani in [20]. The numerical results presented in that paper show the fast convergence of this method, when applied to integral equations.

The present paper is organized as follows. After an introduction to the present work, we review RH functions in Section 2. In Section 3, we consider two-dimensional FIEs of the form

$$u(x,t) = f(x,t) + \int_c^d \int_a^b k(x,t,y,z)G(u(y,z))dy\,dz, \quad (x,t) \in \Omega = [a,b]\times[c,d], \tag{1}$$

where $u(x,t)$ is an unknown real valued function and $f(x,t)$ and $k(x,t,y,z)$ are given continuous functions defined, respectively on Ω and $\Omega\times\Omega$, and $G(u(x,t))$ is a polynomial of $u(x,t)$ with constant coefficients. The aim of this paper is reducing (1) to a nonlinear system of algebraic equations by applying RH functions and bivariate collocation method. For convenience, we assume that

$$G(u(y,z)) = [u(y,z)]^p, \tag{2}$$

where p is a positive integer, but the method can be easily extended and applied to any nonlinear two-dimensional FIE of the form (1), where $G(u(x,t))$ is a polynomial of $u(x,t)$ with constant coefficients. Also, we assume k is such that (1) possesses a unique solution $u(x,t) \in C(\Omega)$.

In Section 4, we explain how the method presented in Section 3 can be applied to nonlinear two-dimensional VIEs of the form

$$u(x,t) = f(x,t) + \int_c^t \int_a^x k(x,t,y,z)G(u(y,z))dy\,dz, \quad (x,t) \in \Omega = [a,b]\times[c,d]. \tag{3}$$

In Section 5, we show that the vector of RH function coefficients of $[u(x,t)]^p$ can be computed in terms of RH function coefficients of $u(x,t)$. In Section 6, we present some numerical examples which show the efficiency and accuracy of the proposed method. Finally, we write the main conclusions of the work in Section 7.

2. Review of RH functions

RH functions have been widely used for solving different problems [17–23]. In this section, we briefly review this class of functions.

Definition 2.1. The Haar wavelet is the function defined on the real line $\mathbb{R}$ as

$$H(x) = \begin{cases} 1, & 0 \leqslant x < \frac{1}{2}, \\ -1, & \frac{1}{2} \leqslant x < 1, \\ 0, & \text{otherwise}. \end{cases} \tag{4}$$