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Numerical solution of nonlinear Volterra integro-differential equations of arbitrary order by CAS wavelets

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ABSTRACT

A computational method for numerical solution of a nonlinear Volterra integro-differential equation of fractional (arbitrary) order which is based on CAS wavelets and BPFs is introduced. The CAS wavelet operational matrix of fractional integration is derived and used to transform the main equation to a system of algebraic equations. Some examples are included to demonstrate the validity and applicability of the technique.

1. Introduction

The aim of this work is to present an operational method (operational CAS wavelet method) for approximating the solution of a nonlinear fractional integro-differential equation of the second kind:

$$D_*^\alpha f(x) - \lambda \int_0^x k(x,t)F(f(t))\,dt = g(x), \quad q > 1 \tag{1.1}$$

with these supplementary conditions:

$$f^{(i)}(0) = \delta_i, \quad i = 0, 1, \ldots, r-1, \quad \ni \; r-1 < \alpha \leqslant r, \quad r \in \mathbb{N}, \tag{1.2}$$

where, $g \in L^2([0,1))$, $k \in L^2([0,1)^2)$ are known functions, $f(x)$ is the unknown function, D_*^α is the Caputo fractional differentiation operator of order α and $F(f(x))$ is a polynomial of $f(x)$ with constant coefficients. For convenience, we put $F(f(x)) = [f(x)]^q$ such that $q \in \mathbb{N}$. Such kind of equations arise in the mathematical modeling of various physical phenomena, such as heat conduction in materials. Moreover, these equations are encountered in combined conduction, convection and radiation problems [1–3]. Local and global existence and uniqueness solution of the integro-differential equation given by (1.1) and (1.2) is given in [4]. In recent years, fractional integro-differential equations have been investigated by many authors [5–10]. Most of the methods have been utilized in linear problems and a few number of works have considered nonlinear problems.

In this paper, we introduce a new operational method to solve nonlinear Volterra integro-differential equation of fractional or nonlinear order. The method is based on reducing the equation to a system of algebraic equations by expanding

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the solution as CAS wavelets with unknown coefficients. The main characteristic of the operational method is to convert a differential equation into an algebraic one. It not only simplifies the problem but also speeds up the computation. It is considerable that, for $\alpha \in \mathbb{N}$, Eqs. (1.1) and (1.2) are ordinary Volterra integro-differential equations and the method can be easily applied for them.

1.1. Fractional calculus

There are several definitions of a fractional derivative of order $\alpha > 0$. The two most commonly used definitions are the Riemann–Liouville and Caputo. Each definition uses Riemann–Liouville fractional integration and derivatives of whole order. The Riemann–Liouville fractional integration of order α is defined as:

$$J^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(x)\,dt, \quad x > 0, \quad J^0 f(x) = f(x)$$

and the Caputo fractional derivatives of order α is defined as $D_*^{\alpha}f(x) = J^{m-\alpha}D^m f(x)$, where D^m is the usual integer differential operator of order m and $J^{m-\alpha}$ is the Riemann–Liouville integral operator of order $m-\alpha$ and $m-1 < \alpha \leqslant m$. The Caputo fractional derivative computes an ordinary derivative followed by a fractional integral to achieve the desired order of fractional derivative. The Riemann–Liouville fractional derivative is computed in the reverse order. Therefore, the Caputo fractional derivative allows traditional initial and boundary conditions to be included in the formulation of the problem, but the Riemann–Liouville fractional derivative allows initial conditions in terms of fractional integrals and their derivatives.

The relation between the Riemann–Liouville operator and Caputo operator is given by the following lemma [14]:

Lemma 1.1. *If $m-1 < \alpha \leqslant m, m \in \mathbb{N}$, then $D_*^{\alpha}J^{\alpha}f(x) = f(x)$, and:*

$$J^{\alpha}D_*^{\alpha}f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!}, \quad x > 0.$$

2. CAS wavelets and block pulse functions (BPFs)

In this section first we give some necessary mathematical preliminaries of CAS wavelets and block pulse functions (BPFs) which are used further in this paper, then function approximation via these two conceptions is introduced.

Wavelets are mathematical functions that are constructed using dilation and translation of a single function called the mother wavelet denoted by $\psi(x)$ and must satisfy certain requirements. If the dilation parameter is a and translation parameter is b then we have the following family of wavelets:

$$\psi_{a,b}(x) = |a|^{-1/2}\psi\left(\frac{x-b}{a}\right), \quad a, b \in \mathbb{R}, \quad a \neq 0. \tag{2.1}$$

Restricting a and b to discrete values such as $a = a_0^{-k}, b = nb_0a_0^{-k}$, $a_0 > 1$, $b_0 > 0$ and n and k are positive integers, gives:

$$\psi_{k,n}(x) = |a_0|^{k/2}\psi\left(a_0^k x - nb_0\right), \tag{2.2}$$

where $\psi_{k,n}(x)$ form a basis for $L^2(\mathbb{R})$. If $a_0 = 2$ and $b_0 = 1$, then it is clear that the set $\{\psi_{k,n}(x)\}$ forms an orthonormal basis for $L^2(\mathbb{R})$.

The CAS wavelets employed in this paper are defined as:

$$\psi_{n,m}(x) = \begin{cases} 2^{k/2} CAS_m(2^k x - n), & \text{if } \frac{n}{2^k} \leqslant x < \frac{n+1}{2^k}; \\ 0, & \text{otherwise}, \end{cases}$$

where:

$$CAS_m(x) = cos(2m\pi x) + sin(2m\pi x)$$

and $n = 0, 1, \ldots, 2^k - 1, k \in \mathbb{N} \cup \{0\}, m \in \mathbb{Z}$.

CAS wavelets have compact support i.e:

$$Supp(\psi_{n,m}(x)) = \overline{\{x : \psi_{n,m}(x) \neq 0\}} = \left[\frac{n}{2^k}, \frac{n+1}{2^k}\right].$$

Let us introduce the following useful notation, corresponding to CAS wavelets:

$$\widetilde{\psi}_{n,m}(x) = \begin{cases} 2^{k/2} CAS_m(n - 2^k x), & \text{if } \frac{n}{2^k} \leqslant x < \frac{n+1}{2^k}; \\ 0, & \text{otherwise}. \end{cases}$$