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Infinitely many positive solutions of a singular dirichlet problem involving the p-Laplacian

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ABSTRACT

By using critical point theory and the method of lower and upper solutions, we obtained the existence of two unbounded sequences of positive solutions, which are, respectively, characterized as local minimizers and saddle points of the relative functional, of a singular dirichlet problem involving the p-Laplacian.

1. Introduction

The p-Laplacian operator appears in many research areas. For instance, in the study of torsional creep (elastic for $p = 2$, plastic as $p \to +\infty$), flow through porous media ($p = \frac{3}{2}$) or glacial sliding ($p \in (1, \frac{4}{3}]$), see [2]. The existence of multiple solutions of the p-Laplacian problem was considered by many papers, see for example [1,3,5,6] and the references therein. Most of them treated the problem under conditions on f which imply some sort of oscillations between a sublinear and a superlinear behaviour.

In [3], a contribution was made for when $\frac{pF(x,s)}{|s|^p}$ interacts asymptotically with the first eigenvalue. In [5], the existence of at least one solution was obtained when the nonlinearity $\frac{pF(x,s)}{|s|^p}$ stays asymptotically between the two first eigenvalues of the p-Laplacian operator. More recently, in [6] the authors obtained the existence of multiple nontrivial solutions for the case $\lim_{s\to+\infty} \frac{pF(x,s)}{|s|^p} < \lambda_1$.

Motivated by above works, in this paper, we consider the following singular problem

$$\begin{cases} (\Phi_p(u'))' + f(t,u) = 0, & t \in (0,1), \\ u(0) = 0 = u(1), \end{cases} \tag{1.1}$$

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where $p > 1$, $\Phi_p(s) = |s|^{p-2}s$ and f may be singular at $u = 0$. Combining critical point theory and the method of lower and upper solutions, we proved that problem (1.1) has two unbounded sequences of positive solutions, which are, respectively, characterized as local minimizers and saddle points of the relative functional, assuming that

$$\lim_{s\to+\infty} f(t_0, s) = +\infty \text{ and } \lim\ \sup_{s\to+\infty} \frac{pF(t,s)}{s^p} > \left(\frac{4p}{r(p-1)}\right)^p,$$

with some uniformity in an open neighborhood $\cup(t_0, r) \subset (0,1)$, and

$$\lim\ \inf_{s\to+\infty} \frac{pF(t,s)}{s^p} < \frac{p^{p-1}}{(p-1)^p},$$

with some uniformity in (0,1), where $F(t,s) = \int_c^s f(t,\tau)d\tau$ and $c > 0$.

This work is organized as follows. In Section 2, some notations and preliminaries are introduced. The existence of infinitely many positive solutions of problem (1.1) is discussed in Section 3. As applications of our results, an example is given in the last section.

2. Preliminary

Consider the problem

$$\begin{cases} (\Phi_p(u'))' + \varphi(t) = 0, & t \in (0,1), \\ u(0) = 0 = u(1), \end{cases} \tag{2.1}$$

where $\varphi : [0,1] \to [0,+\infty)$ is continuous. We know that problem (2.1) has a unique positive solution $u \in C[0,1]$ which can be expressed in the form

$$u(t) = \begin{cases} \int_0^t \Phi_p^{-1}\left(\int_s^\sigma \varphi(\tau)d\tau\right)ds, & 0 \leqslant t \leqslant \sigma, \\ \int_t^1 \Phi_p^{-1}\left(\int_\sigma^s \varphi(\tau)d\tau\right)ds, & \sigma \leqslant t \leqslant 1, \end{cases} \tag{2.2}$$

where σ satisfies the equation

$$\int_0^t \Phi_p^{-1}\left(\int_s^t \varphi(\tau)d\tau\right)ds = \int_t^1 \Phi_p^{-1}\left(\int_t^s \varphi(\tau)d\tau\right)ds, \quad 0 < t < 1.$$

Then for $\epsilon > 0$, problem

$$\begin{cases} (\Phi_p(u'))' + 2\epsilon = 0, & t \in (0,1), \\ u(0) = 0 = u(1), \end{cases} \tag{2.3}$$

has a uniqueness solution $y_\epsilon(t) \in C[0,1]$ which can be expressed in the form

$$y_\epsilon(t) = \begin{cases} \int_0^t \Phi_p^{-1}(\epsilon(1-2s))ds, & 0 \leqslant t \leqslant \frac{1}{2}, \\ \int_t^1 \Phi_p^{-1}(\epsilon(2s-1))ds, & \frac{1}{2} \leqslant t \leqslant 1. \end{cases} \tag{2.4}$$

Consider the problem

$$\begin{cases} (\Phi_p(u'))' + g(t,u) = 0, & t \in (0,1), \\ u(0) = 0 = u(1), \end{cases} \tag{2.5}$$

where $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function. Fixed $c > 0$, define a C^1 functional $\Phi : W_0^{1,p}(0,1) \to \mathbb{R}$ by

$$\Phi(u) = \int_0^1 \left[\frac{|u'(t)|^p}{p} - G(t,u(t))\right]dt, \tag{2.6}$$

where $G(t,u) = \int_c^u g(t,s)ds$ and $W_0^{1,p}(0,1)$ is the usual Sobolev space, normed by

$$||u|| = \left(\int_0^1 |u'(t)|^p dt\right)^{\frac{1}{p}}. \tag{2.7}$$

Definition 2.1. A function $\alpha : [0,1] \to \mathbb{R}$ is a lower solution of (2.5) if $\alpha \in C^1[0,1]$, $\Phi_p(\alpha\prime) \in AC[0,1]$ such that

$$(\Phi_p(\alpha'(t)))' + g(t,\alpha(t)) \geqslant 0, \quad \text{a.e. } t \in (0,1),$$
$$\alpha(0) \leqslant 0, \quad \alpha(1) \leqslant 0.$$