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Quasi-periodic solutions and periodic bursters in quasiperiodically driven oscillators

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ABSTRACT

In this paper, we propose a perturbation method to determine an approximation and conditions of existence of quasi-periodic (QP) solutions and bursting dynamics in a quasi-periodically driven system. The QP forcing consists of two periodic excitations, one with a very slow frequency and the other with a frequency of the same order of the proper frequency of the oscillator. A first averaging is done over the fast dynamics, then the quasi-static solutions of the modulation equations of amplitude and phase are determined and their stability analyzed. We show that a necessary condition for the occurrence of periodic bursters is that the slow excitation is parametric.

1. Introduction

Quasi-periodically forced systems are those that are influenced by two periodic signals with incommensurate frequencies. A special interest has been given to the construction and the existence of regular motions through numerical and perturbation methods. Broer and Simó [1] geometrically explored resonance tongues containing instability pockets in a linear Hill's equation with quasi-periodic forcing. Rand et al. [2] determined an approximation of the regions of stability using Lyapunov exponents and the harmonic balance method. Belhaq and co-workers [3] approximated analytically QP solutions of a damped cubic nonlinear QP Mathieu equation, using the double perturbation method. This method uses two perturbation parameters to make natural the application of two reductions through perturbation methods [4]. This method was used when one of the two frequencies is of order $\mathcal{O}(\varepsilon)$.

In this paper, we study QP excitations consisting of a frequency ν of the same order of the proper frequency of the oscillator and a *very slow* frequency ε^p where p is an integer greater than 1. This very slow excitation induces quasi-static solutions on the slow manifold resulting from an averaging over the fast scale of time. Consequently, a change in the nature of the quasi-static solutions during a period of the very slow frequency, can lead to the appearance of periodic bursters. The resonant and non-resonant cases, as well as the external and parametric excitations, are discussed. Thus, we focus in this work on the conditions of existence and the construction of QP solutions and periodic bursters solutions. For detailed classification of bursters, see Golubitsky et al. [5].

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This paper is organized as follows: in Section 2 we state the main results related to the existence of QP solutions and periodic bursters using an average over the fast dynamic. In Section 3, two examples are given. The first is an oscillator with both parametric damping and nonlinear damping. The second example is a quasi-periodically excited van der Pol oscillator. In Section 4, a summary of the results is given.

2. Formulation of the method

We consider the following quasi-periodically driven system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \varepsilon \mathbf{g}(\nu t, \tau, \mathbf{x}; \mu), \tag{1}$$

where $\mathbf{x} \in \mathrm{IR}^2$, $\mathbf{f}(\mathbf{x})$ is a linear function of $\mathbf{x}$, $\tau = \varepsilon^p t$ with $p \geqslant 2$ and ε is a small positive parameter, μ represents the set of parameters of the system and $\mathbf{g}$ is polynomial in $\mathbf{x}$ and 2π-periodic in νt and τ. The frequency ν has the same order as the proper frequency of the system, that is assumed, to be equal to unity. We say that the function $\mathbf{g}$ is a QP function with basic frequencies ν and ε^p i.e., it contains the terms $\cos(\nu t)$ and $\cos(\varepsilon^p t)$.

We assume that for $\varepsilon = 0$ Eq. (1) has an elliptic equilibrium $\mathbf{x} = \mathbf{0}$. In what follows, the multiple scales method (MSM) is used [6] to construct an approximation of solutions of Eq. (1). Thus, the solution $\mathbf{x}(t, \varepsilon)$ of Eq. (1) and the independent scales of time T_i are expressed as follows:

$$\mathbf{x}(t; \varepsilon) = \sum_{i=0}^{N} \varepsilon^i \mathbf{x}_i(T_0, T_1, \ldots, T_N) + \mathcal{O}(\varepsilon^{N+1}), \tag{2}$$

$$T_i = \varepsilon^i t, \frac{\mathrm{d}}{\mathrm{d}t} = \sum_{i=0}^{N} \varepsilon^i D_i, \quad \text{where} \quad D_i = \frac{\partial}{\partial T_i} \tag{3}$$

The functions $\mathbf{x}_i$ are assumed to be periodic. Substituting Eqs. (2) and (3) into the original Eq. (1), we obtain the following hierarchy of problems:

$$\text{order } \mathcal{O}(1) : D_0 \mathbf{x}_0 = \mathbf{f}(\mathbf{x}_0). \tag{4}$$

The unperturbed solution $\mathbf{x}_0$ is assumed, without loss of generality, to be periodic with the frequency 1.

$$\text{order } \mathcal{O}(\varepsilon) : D_0 \mathbf{x}_1 = -D_1 \mathbf{x}_0 + \mathbf{g}(\nu t, \tau, \mathbf{x}_0; \mu) + \frac{1}{\varepsilon}[\mathbf{f}(\mathbf{x}_0 + \varepsilon \mathbf{x}_1) - \mathbf{f}(\mathbf{x}_0)]. \tag{5}$$

The usual approach in studying such systems is based on the so-called quasi-steady state assumption which means that the fast variables are in a quasi-steady state; i.e., the fixed points are no more static points but they depend on the very slow excitation $\cos(\varepsilon^p t)$.

At this level of computations one should discuss the existence of resonances between the forcing $\cos(\nu t)$ and the unperturbed frequency.

2.1. Non-resonant case

The elimination condition of secular terms, from Eq. (5), can be written as

$$D_1 \mathbf{x}_0 = \mathbf{g}^*(\tau, \mathbf{x}_0; \mu), \tag{6}$$

where the function $\mathbf{g}^*$ contains the contribution of odd terms with respect to $\mathbf{x}$ contained in the function $\mathbf{g}$. The number of the zeros of $\mathbf{g}^*(\tau, \mathbf{x}_0; \mu)$ determines the number of the solutions $\mathbf{x}$ to Eq. (1) that coexist for the same values of μ.

Let $\mathbf{X}_0$ be a solution of $\mathbf{g}^*(\tau, \mathbf{X}_0; \mu) = 0$. The function $\mathbf{g}^*$ can depend on the slow time scale τ only when the slow excitation $\mathbf{g}$ contains a parametric term of the form $\cos(\varepsilon^p t) x^{2n+1}$ with $n \in \mathrm{IN}$. In this case, the solution $\mathbf{X}_0$ depends on the slow time scale and the set of parameters μ i.e., $\mathbf{X}_0(\tau; \mu)$. Hence, during a period of the slow time scale $2\pi/\varepsilon^p$, the solution $\mathbf{X}_0(\tau; \mu)$ can lose stability or disappear. When these processes lead to the stabilisation or the appearance of another bounded solution, the system (1) will have a periodic burster solution.

In the case where the slow excitation is external, the function $\mathbf{g}^*(\mathbf{x}_0; \mu)$ does not depend on the slow time scale τ and consequently its zeros $\mathbf{X}_0(\mu)$ also. Thus, the existence of periodic bursters is excluded in this case.

In the case of parametric slow excitation of the form $\cos(\varepsilon^p t)\mathbf{x}^{2n}$, the approximated solutions $\mathbf{x}$ Eq. (1) is three-period-QP with three fundamental frequencies 1, ν and ε^p.

In the case of parametric slow excitation of the form $\cos(\varepsilon^p t)\mathbf{x}^{2n+1}$, the approximated solution of $\mathbf{x}$ of Eq. (1) can be a periodic burster. Otherwise it is a phase slowly modulated solution.

2.2. Resonant case

Here we restrict our study to the case where $\nu = m + \varepsilon\sigma$ with $m \in \mathrm{IN}^*$ and σ is a detuning parameter. The condition of elimination of secular terms can be written as follows: