Available online at www.sciencedirect.com
ScienceDirect
Communications in Nonlinear Science and Numerical Simulation
Communications in Nonlinear Science and Numerical Simulation 14 (2009) 674–684
www.elsevier.com/locate/cnsns

Homotopy analysis method for fractional IVPs

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Received 17 September 2007; accepted 30 September 2007
Available online 16 October 2007

Abstract

In this paper, the homotopy analysis method is applied to solve linear and nonlinear fractional initial-value problems (fIVPs). The fractional derivatives are described by Caputo's sense. Exact and/or approximate analytical solutions of the fIVPs are obtained. The results of applying this procedure to the studied cases show the high accuracy and efficiency of the approach.

PACS: 02.30.Hq; 02.30.Mv; 02.60.Cb

Keywords: Fractional IVPs; Homotopy analysis method; Caputo's fractional derivative

1. Introduction

In recent years, fractional differential equations (FDEs) have found applications in many problems in physics and engineering [1–4]. Since most of the nonlinear FDEs cannot be solved exactly, approximate and numerical methods must be used. Some of the recent analytic methods for solving nonlinear problems include the Adomian decomposition method (ADM) [5–9], homotopy-perturbation method (HPM) [10,11], variational iteration method (VIM) [12,13] and homotopy analysis method (HAM) [14–19]. The HAM, first proposed in 1992 by Liao [14], has been successfully applied to solve many problems in physics and science [20–29]. Very recently, Song and Zhang [30] applied HAM to solve fractional KdV–Burgers–Kuramoto equation.

In this paper, HAM is applied to solve linear and nonlinear fractional initial-value problems (fIVPs). Some test examples shall be presented to show the efficiency and accuracy of HAM. Furthermore, the Taylor series expansion shall be employed to avoid the difficulties with radical nonlinear terms.

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1007-5704/$ - see front matter
doi:10.1016/j.cnsns.2007.09.014

2. Basic definitions

In this section, we give some definitions and properties of the fractional calculus [4].

Definition 1. A real function $h(t), t > 0$, is said to be in the space $C_\mu, \mu \in R$, if there exists a real number $p > \mu$, such that $h(t) = t^p h_1(t)$, where $h_1(t) \in C(0,\infty)$, and it is said to be in the space C_μ^n if and only if $h^{(n)} \in C_\mu, n \in N$.

Definition 2. The Riemann–Liouville fractional integral operator (J^α) of order $\alpha \geqslant 0$, of a function $h \in C_\mu, \mu \geqslant -1$, is defined as

$$J^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} h(\tau) \mathrm{d}\tau \ (\alpha > 0),$$
$$J^0 h(t) = h(t) \tag{1}$$

$\Gamma(z)$ is the well-known Gamma function. Some of the properties of the operator J^α, which we will need here, are as follows:

For $h \in C_\mu$, $\mu \geqslant -1$, $\alpha, \beta \geqslant 0$ and $\gamma \geqslant -1$:

(1) $J^\alpha J^\beta h(t) = J^{\alpha+\beta} h(t)$,
(2) $J^\alpha J^\beta h(t) = J^\beta J^\alpha h(t)$,
(3) $J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

Definition 3. The fractional derivative (D^α) of $h(t)$ in the Caputo's sense is defined as

$$D^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} h^{(n)}(\tau) \mathrm{d}\tau, \tag{2}$$

for $n-1 < \alpha \leqslant n, \ n \in N, \ t > 0, \ h \in C_{-1}^n$.

The following are two basic properties of the Caputo's fractional derivative [31]:

(1) Let $h \in C_{-1}^n, \ n \in N$. Then $D^\alpha h, \ 0 \leqslant \alpha \leqslant n$ is well defined and $D^\alpha h \in C_{-1}$.
(2) Let $n-1 < \alpha \leqslant n, \ n \in N$ and $h \in C_\mu^n, \ \mu \geqslant -1$. Then

$$(J^\alpha D^\alpha) h(t) = h(t) - \sum_{k=0}^{n-1} h^{(k)}(0^+) \frac{t^k}{k!}. \tag{3}$$

3. The homotopy analysis method (HAM)

In HAM, FDEs are written in the form,

$$\mathscr{F}\mathscr{D}(u(t)) = 0, \tag{4}$$

where $\mathscr{F}\mathscr{D}$ is a fractional differential operator, t denotes an independent operator and $u(t)$ is an unknown function.

In the frame of HAM [18,19], we can construct the following zeroth-order deformation:

$$(1-q)\mathscr{L}(U(t;q) - u_0(t)) = q\hbar H(t) \mathscr{F}\mathscr{D}(U(t;q)), \tag{5}$$

where $q \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is an auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, $\mathscr{L}$ is an auxiliary linear operator, $u_0(t)$ is an initial guess of $u(t)$ and $U(t;q)$ is an unknown function on the independent variables t and q.

Obviously, when $q = 0$ and $q = 1$, it holds

$$U(t;0) = u_0(t), \quad U(t;1) = u(t), \tag{6}$$