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Applying a new algorithm to derive nonclassical symmetries

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Abstract

In this paper we extend the procedure described for Bîlă and Niesen in [Bîlă N, Niesen J. On a new procedure for finding nonclassical symmetries. J Symbol Comp 2004;38:1523–33], to obtain the determining equations of the nonclassical symmetries associated with a partial differential equation system, to a different case. We offer some examples of how our method works. By using this procedure we obtain a new nonclassical symmetry for the $2+1$-dimensional shallow water wave equation.

Keywords: Partial differential equations; Symmetries; Nonclassical method

1. Introduction

The application of Lie transformations group theory for the construction of solutions of nonlinear partial differential equations (PDEs) is one of the most active fields of research in the theory of nonlinear PDEs and applications.

Motivated by the fact that symmetry reductions for many PDEs are known that are not obtained by using the classical symmetries, there have been several generalizations of the classical Lie group method for symmetry reductions. The notion of nonclassical symmetries was firstly introduced by Bluman and Cole [1] to study the symmetry reductions of the heat equation. The description of the method can be found in [1,3,7]. In [4] Clarkson and Mansfield proposed an algorithm for calculating the determining equations associated with the nonclassical method: the PDE system is augmented with the invariant surface conditions, the nonclassical symmetries are found by seeking the classical symmetries of the augmented system while demanding that the symmetries operator be related to the invariant surface condition.

Bîlă and Niesen in [2] dropped this requirement. Their procedure consists in reducing the augmented PDE system to its involutive form and then applying the classical Lie method to the reduced PDE system, but with an arbitrary symmetry operator which is not related anymore to the invariant surface condition.

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In this paper we extend the procedure described in [2] to a different case. We apply the procedure to a Cahn–Hilliard equation, to a Boussinesq equation and to a $2+1$-dimensional shallow water wave equation.

2. Nonclassical symmetries

Bluman and Cole [1], in their study of symmetry reductions of the heat equation, proposed the called nonclassical method. The basic idea of the method applied to the general nth order PDE, with p independent variables, $x = (x_1, \ldots, x_p)$, and one dependent variables, $u = u(x)$,

$$\Delta \equiv \Delta(x, u, \mathbf{u}^{(l)}(x), \ldots, \mathbf{u}^{(n)}(x)) = 0, \tag{1}$$

where $\mathbf{u}^{(l)}(x)$ denotes the set of all the partial derivatives of order l of u, is the following:

The PDE (1) is augmented with the invariance surface condition

$$\Psi \equiv \sum_{i=1}^{p} \xi_i(x, u) \frac{\partial u}{\partial x_i} - \eta(x, u) = 0, \tag{2}$$

which is associated with the vector field

$$V = \sum_{i=1}^{p} \xi_i(x, u) \frac{\partial}{\partial x_i} + \eta(x, u) \frac{\partial}{\partial u}. \tag{3}$$

Let us consider the submanifold

$$S_\Delta = \{u(x) : \Delta = 0\}, \tag{4}$$

i.e., the set of solutions of the system (1). In the nonclassical method one requires that the subset of S_Δ given by

$$S_{\Delta,\Psi} = \{u(x) : \Delta = 0, \Psi = 0\}, \tag{5}$$

are invariant under the transformation with infinitesimal generator (3).

The application of the criterion for infinitesimal invariance to the equation (1) and the invariant surface condition (2) require that

$$\mathrm{pr}^{(n)} V(\Delta)_{\Delta=0,\Psi=0} = 0, \qquad \mathrm{pr}^{(n)} V(\Psi)_{\Delta=0,\Psi=0} = 0, \tag{6}$$

and we obtain an overdetermined nonlinear system of equations for the infinitesimals. The number of determining equations arising in the nonclassical method is smaller than for the classical method, consequently the set of solutions is, in general, larger than for the classical method [4].

• If $\xi_p \neq 0$ Bîlă and Niesen proposed an algorithm for finding nonclassical symmetries which is based on the following procedure: Since if V is a vector field then so is λV, for any function $\lambda = \lambda(x, u)$, if $\xi_p \neq 0$ we can multiply V for $\frac{1}{\xi_p}$ and the invariant surface conditions is,

$$\frac{\partial u}{\partial x_p} = \eta(x, u) - \sum_{i=1}^{p-1} \xi_i(x, u) \frac{\partial u}{\partial x_i}. \tag{7}$$

Substituting (7) and its derivatives with respect to x in (1) we obtain a new PDE

$$\Delta' \equiv \Delta'(\mathscr{A}_v(x, u), \mathbf{u}^{[l]}, \ldots, \mathbf{u}^{[n]}) = 0, \tag{8}$$

for the unknown function $u = u(x_1, \ldots, x_{p-1};\, x_p)$ of $x_1, \ldots, x_{p-1}$ (here x_p is considered as a parameter); where $\mathscr{A}_v(x, u)$ are the coefficients of $u^{[l]}$, and $u^{[l]}$ denotes the set of all the partial derivatives of u with respect to $x = (x_1, x_2, \ldots, x_{p-1})$ up to order N. Applying the classical Lie method to (8), if (8) is of maximal rank. Invariance of (8) under a Lie group of point transformation, with infinitesimal generator

$$W = \sum_{i=1}^{p} s_i(x, u) \frac{\partial}{\partial x_i} + r(x, u) \frac{\partial}{\partial u}, \tag{9}$$