Available online at www.sciencedirect.com
ScienceDirect
Communications in Nonlinear Science and Numerical Simulation
ELSEVIER
Communications in Nonlinear Science and Numerical Simulation 13 (2008) 584–592
www.elsevier.com/locate/cnsns

The tanh method for travelling wave solutions to the Zhiber–Shabat equation and other related equations

Abdul-Majid Wazwaz *

Department of Mathematics and Computer Science, Saint Xavier University, 3700 W, 103rd Street, Chicago, IL 60655, United States

Received 13 May 2006; received in revised form 29 June 2006; accepted 29 June 2006
Available online 24 August 2006

Abstract

The tanh method and the extended tanh method are used for handling the Zhiber–Shabat equation and the related equations: Liouville equation, sinh-Gordon equation, Dodd–Bullough–Mikhailov (DBM) equation, and Tzitzeica–Dodd–Bullough equation. Travelling wave solutions of different physical structures are formally derived for each equation.

PACS: 02.30.Ik; 03.40.Kf

Keywords: Zhiber–Shabat equation; Liouville equation; Sinh-Gordon equation; Dodd–Bullough–Mikhailov equation; Tzitzeica–Dodd–Bullough equation; The tanh method

1. Introduction

In this work, we investigate the nonlinear Zhiber–Shabat equation [1–9]:

$$u_{xt} + p\mathrm{e}^{u} + q\mathrm{e}^{-u} + r\mathrm{e}^{-2u} = 0, \tag{1}$$

where p, q, and r are arbitrary constant. For $q = r = 0$, Eq. (1) reduces to the Liouville equation. For $r = 0$, Eq. (1) reduces to the sinh-Gordon equation. However, for $q = 0$, Eq. (1) gives the well-known Dodd–Bullough–Mikhailov equation. Moreover, for $p = 0$, $q = -1$, $r = 1$, we obtain the Tzitzeica–Dodd–Bullough equation. The aforementioned equations play a significant role in many scientific applications such as solid state physics, nonlinear optics, plasma physics, fluid dynamics, mathematical biology, nonlinear optics, dislocations in crystals, kink dynamics, and chemical kinetics, and quantum field theory [1–9].

Searching for explicit solutions for nonlinear evolution equation, by using different methods, is the goal for many researchers. Many powerful methods, such as Bäcklund transformation, inverse scattering method [8], Hirota bilinear forms, pseudo spectral method, the tanh–sech method [10–14], the sine–cosine method [15–20], projective Riccati equation method [7], and many other techniques were used to investigate these types of

* Tel.: +1 773 298 3397; fax: +1 773 779 9061.
E-mail address: wazwaz@sxu.edu

1007-5704/$ - see front matter
doi:10.1016/j.cnsns.2006.06.014

equations and to derive characteristics of the obtained solutions. Practically, there is no unified method that can be used to handle all types of nonlinear problems.

The present paper is motivated by the desire to extend the works in [13–15] to make further progress. The tanh method, developed by Malfliet in [10], proved to be effective and reliable for several nonlinear problems. This method works effectively if the equation involves sine, cosine, hyperbolic sine, hyperbolic cosine functions, and exponential functions. The extended tanh method will be employed as well. In what follows we highlight the main features of the tanh method as introduced in [10] where more details and examples can be found there.

2. The tanh method and the extended tanh method

We first unite the independent variables x and t into one wave variable $\xi = x - ct$ to carry out a PDE in two independent variables:

$$P(u, u_t, u_x, u_{xx}, u_{xxx}, \ldots) = 0, \tag{2}$$

into an ODE

$$Q(u, u', u'', u''', \ldots) = 0. \tag{3}$$

Eq. (3) is then integrated as long as all terms contain derivatives. The tanh technique is based on the *a priori* assumption that the traveling wave solutions can be expressed in terms of the tanh function [10]. We therefor introduce a new independent variable

$$Y = \tanh(\mu\xi), \tag{4}$$

that leads to the change of derivatives:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\xi} &= \mu\big(1 - Y^2\big)\frac{\mathrm{d}}{\mathrm{d}Y}, \\ \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} &= \mu^2(1 - Y^2)\left(-2Y\frac{\mathrm{d}}{\mathrm{d}Y} + (1 - Y^2)\frac{\mathrm{d}^2}{\mathrm{d}Y^2}\right). \end{aligned} \tag{5}$$

The solutions can be proposed by the tanh method as a finite power series in Y in the form:

$$u(\mu\xi) = S(Y) = \sum_{k=0}^{M} a_k Y^k, \tag{6}$$

limiting them to solitary and shock wave profiles. However, the extended tanh method admits the use of the finite expansion

$$u(\mu\xi) = S(Y) = \sum_{k=0}^{M} a_k Y^k + \sum_{k=1}^{M} b_k Y^{-k}, \tag{7}$$

where M is a positive integer, in most cases, that will be determined. Expansion (7) reduces to the standard tanh method [10–12] for $b_k = 0$, $1 \leqslant k \leqslant M$. Substituting (6) or (7) into the ODE (3) results in an algebraic equation in powers of Y.

The parameter M is a positive integer, in most cases, that will be determined by using a balance procedure, where by comparing the behavior of Y^M in the highest derivative against its counterpart within the nonlinear terms. With M determined, we collect all coefficients of powers of Y in the resulting equation where these coefficients have to vanish, hence the coefficients a_k, $k \geqslant 0$ can be determined.

3. The Zhiber–Shabat equation

We first use $u(x,t) = u(\xi)$ that will carry out the Zhiber–Shabat Eq. (1) into

$$-cu_{\xi\xi} + p\mathrm{e}^{u} + q\mathrm{e}^{-u} + r\mathrm{e}^{-2u} = 0. \tag{8}$$