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# Cosmological simulations of structure formation and the Vlasov equation

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**Abstract**

In cosmology numerical simulations of structure formation are now of central importance, as they are the sole instrument for providing detailed predictions of current cosmological models for a whole class of important constraining observations. These simulations are essentially molecular dynamics simulations of $N$ ($\gg$1, now up to of order several billion) particles interacting through their self-gravity. While their aim is to produce the Vlasov limit, which describes the underlying (‘‘cold dark matter’’) models, the degree to which they actually do produce this limit is currently understood, at best, only very qualitatively, and there is an acknowledged need for ‘‘a theory of discreteness errors’’. In this talk I will describe, for non-cosmologists, both the simulations and the underlying theoretical models, and will then focus on the issue of discreteness, describing some recent progress in addressing this question quantitatively.


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## 1. Introduction

Cosmology is a branch of physics which has been extremely dynamic in the last decade or so. This dynamism has come essentially from the explosion of new data constraining cosmological models, from a whole range of observational techniques. Most important of all are, without doubt, the observations of the ‘‘three degree Kelvin’’ cosmic microwave background radiation (CMBR): there now exist detailed maps of the tiny fluctuations in the temperature of this radiation as a function of direction on the sky, at an angular resolution well below a degree with a sensitivity of less than 100 μK [20]. This is the ‘‘relic’’ radiation which decoupled (electromagnetically) from matter when the universe was a few hundred thousand years old. These maps of temperature fluctuations thus provide very precise constraints on the spatial fluctuations in the energy density in the universe at this time, which may in turn be related (through the Poisson equation, suitably modified in the context of general relativity) to the fluctuations in the gravitational potential at this time. In the paradigm

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defined by current cosmological models these very low amplitude fluctuations at ''early'' times develop by gravitational instability, giving rise ultimately to the highly inhomogeneous universe observed today: the distribution of galaxies and galaxy clusters which shows a high degree of inhomogeneity with structures extending over a significant fraction of the visible universe (see, e.g., [9]). This is the problem of ''large scale structure formation'', one of the central challenges in cosmology today.

The specific model of the universe which cosmologists have developed to account for current observations posits that the dominant fraction (more than eighty per cent) of the gravitating matter in the universe is so-called ''cold dark matter'' (CDM). This is matter with small velocity dispersion (and therefore non-relativistic) which interacts only gravitationally (or almost only). Further on the scales of interest the approximation of Newtonian gravity is very good. Thus the problem of structure formation can be reduced, to a good approximation, to that of self-gravitating particles in an infinite space evolving from some given initial conditions (see below).[1] These initial conditions (constrained as we have noted by the CMBR) are what we can call quasi-uniform, i.e., they exhibit small deviations from an exactly uniform distribution of constant density, fluctuations averaged in a sphere about the mean density typically decaying away as some power of the radius.

The CDM particles are, further, usually posited to be microscopic, with masses of the order of those of particles in the standard model of particles physics (i.e., within a few orders of magnitude of the proton mass). The macroscopic scales on which fluctuations are probed by cosmological observations thus contain a huge number of such particles (e.g., of order $10^{70}$ to $10^{80}$ in standard models). This suggests that the relevant dynamics of the CDM can be well approximated by the Vlasov limit, taking the particle number $N$ to infinity in an appropriate manner. In fact it is assumed canonically in cosmology that the correct dynamical description of CDM is given by the coupled Vlasov–Poisson equations:

$$\frac{\partial f(\mathbf{r},\mathbf{v},t)}{\partial t} + \mathbf{v}\cdot\nabla f(\mathbf{r},\mathbf{v},t) - \nabla\Phi\cdot\frac{\partial f(\mathbf{r},\mathbf{v},t)}{\partial \mathbf{v}} = 0 \tag{1a}$$

$$\nabla^2\Phi(\mathbf{r},t) = 4\pi G\left[\int d^3v f(\mathbf{r},\mathbf{v},t) - \rho_0\right] \tag{1b}$$

where $f(\mathbf{r},\mathbf{v},t)$ is the smooth one-particle phase space density (i.e., the mass per unit phase space volume), $G$ the gravitational constant, $\Phi$ the gravitational potential and $\rho_0$ the mean mass density.

In the literature on cosmology and astrophysics the use of the Vlasov–Poisson system is usually very qualitatively justified using simple considerations about time scales for two-body collisions (see, e.g., [4]). Authors who give derivations mostly proceed (e.g., [15,17]) through a truncation of the BBGKY hierarchy for the full $N$-particle phase space density, but one by [5] is closely inspired by the approach (see, e.g., [19]) probably more familiar to the audience of this conference, which uses a coarse-graining of the exact single particle ''spiky'' phase space density which obeys a Liouville equation. These derivations of the Vlasov–Poisson limit for the purely self-gravitating CDM system in this context are not rigorous: they show that Eq. (1) may be obtained in a systematic manner when an infinite number of other terms are negligible. They do not show rigorously, however, that the corresponding regime actually exists, nor that it is the one describing the CDM limit. We will not address this question here which may, however, be of interest to participants at this conference.[2]

A central theoretical problem in cosmology is thus the solution of this set of Vlasov–Poisson equations starting from given initial conditions. Analytically they have so far proved intractable albeit for some very trivial cases. Using various perturbative approaches (see, e.g., [15]), however, simple general results can be derived. The most important such result is that, at leading order in a linearisation of the density and velocity perturbation fields, the amplitude of the fluctuations in the density field, analysed in Fourier space, grow in a simple way, *independent of the wavelength*, as a function of time.[3] This approach can be extended to higher order, but breaks down inevitably when the density fluctuations become large.

[1] The expansion of the background space-time, which is taken into account in cosmological simulations, leads only to a very simple modification (see below) of the Newtonian equations of motion in a static space-time.

[2] See contribution of M. Kiessling.

[3] See the contribution of B. Marcos for more detail.