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Anisotropic effective moduli of microcracked materials under antiplane loading

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ABSTRACT

This study focuses on the prediction of the anisotropic effective elastic moduli of a solid containing microcracks with an arbitrary degree of alignment by using the generalized self-consistent method (GSCM). The effective elastic moduli pertaining to anti-plane shear deformation are discussed in detail. The undamaged solid can be isotropic as well as anisotropic. When the undamaged solid is isotropic, the GSCM can be realized exactly. When the undamaged solid is anisotropic it is difficult to provide an analytical solution for the crack opening displacement to be used in the GSCM, thus an approximation of the GSCM is pursued in this case. The explicit expressions of coupled nonlinear equations for the unknown effective moduli are obtained. The coupled nonlinear equations are easily solved through iteration.

1. Introduction

Microcracks are common defects in solids and multiple microcracks usually coexist in a single solid. The prediction of the effective elastic properties of a microcracked solid is technically challenging and can find many practical applications. The major methods developed so far to predict the effective elastic moduli of a microcracked solid include the following: the self-consistent method (SCM) in which a crack is embedded directly into an effective medium [1]; the generalized self-consistent method (GSCM) in which a crack is surrounded by an undamaged matrix region, and then embedded in the effective medium [2,3]; the Mori-Tanaka method (MTM) [4]; the differential scheme method (DS) [5–7]; and the modified differential scheme (MDS) [8]. Recently the GSCM in conjunction with a finite element method (FEM) was developed to take into account crack face contact and friction [9]. Most recently the representative unit cell approach was proposed by Kushch et al. [10] to calculate effective elastic properties of a microcracked solid. Here it shall be noted that the effect of crack orientation statistics on the anisotropic effective moduli of a microcracked solid was in fact first investigated by Santare et al. [3] through the introduction of the crack orientation distribution function $\phi(\theta)$ which was later adopted in [11] within the framework of the SCM to study the problem of cracks with an arbitrarily degree of alignment in a material that is originally anisotropic before the damage occurs.

In this research we analytically study the anisotropic effective elastic moduli of a solid containing microcracks with an arbitrary degree of alignment by using the GSCM. Our model is in principle based on the GSCM developed by Santare et al. [3]. In [3] the GSCM was used to predict the anisotropic moduli under plane stress loading. Here we are concerned with the effective elastic moduli pertaining to anti-plane shear deformation. In our model the undamaged material can be isotropic as well as anisotropic. An exact solution to the cracked elliptical inclusion problem, which is essential in the realization of

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GSCM, can be derived when the undamaged material is isotropic. On the other hand an approximate analytical solution can still be derived when the undamaged material is anisotropic and when the crack density is not very high.

2. The effective moduli of a microcracked solid

For a microcracked solid, the strain energy relationship between the effective medium and the actual microcracked solid can be expressed as [2,3]

$$\frac{1}{2}S^*_{ijkl}\sigma^0_{kl}\sigma^0_{ij} = \frac{1}{2}S_{ijkl}\sigma^0_{kl}\sigma^0_{ij} + \frac{1}{2V}\sum_{k=1}^{M}\int_{C_k}[u_i]t^0_i \mathrm{d}C_k, \tag{1}$$

where S^*_{ijkl} is the effective compliance of the damaged material, S_{ijkl} is the compliance of the undamaged material and V is the sample volume. The above integral is the energy dissipated through the opening of each microcrack, summed over all M cracks, and σ^0_{ij} is the applied homogeneous stress field while $t^0_i = \sigma^0_{ij}n_j$ is the traction along the crack face if the crack did not exist and $[u_i]$ is the crack opening displacement. In this research we focus on the two dimensional case in which all the cracks penetrate the solid through the x_3-direction. In addition we only discuss the effective elastic moduli pertaining to anti-plane shear deformation. In the following we will address two cases: (i) the undamaged material is isotropic; (ii) the undamaged material is anisotropic.

2.1. Isotropic undamaged material

The degree of crack alignment can be described by the crack orientation distribution function $\phi(\theta)$ with θ, ($|\theta| < \pi/2$) being the angle between an individual crack and the positive x_1-axis [3]. Without losing generality, $\phi(\theta)$ can be taken as an even function of θ since we have assumed that the undamaged solid is isotropic. For simplicity, $\phi(\theta)$ is specifically given by [3,9]

$$\phi(\theta) = \begin{cases} \frac{1}{2\theta_0}, & |\theta| \le \theta_0 \\ 0, & |\theta| > \theta_0 \end{cases} \tag{2}$$

where $\theta_0 \leqslant \pi/2$. The two special cases of perfectly aligned cracks and randomly oriented cracks correspond to $\theta_0 = 0$ and $\theta_0 = \pi/2$, respectively, in Eq. (2).

Once we have introduced $\phi(\theta)$, the summation in the energy relationship Eq. (1) can be written as an integral over orientation angle θ,

$$\frac{1}{2}S^*_{ijkl}\sigma^0_{kl}\sigma^0_{ij} = \frac{1}{2}S_{ijkl}\sigma^0_{kl}\sigma^0_{ij} + \frac{M}{2A}\int_{-\theta_0}^{\theta_0}\phi(\theta)\int_{C_k}[u_i]t^0_i \mathrm{d}C_k \mathrm{d}\theta, \tag{3}$$

where A is a representative area of the sample.

In this study, we assume that the material is under anti-plane shear deformation. As a result the above energy relationship Eq. (3) can be simplified as

$$\frac{(\sigma^0_{31})^2}{2C^*_{55}} + \frac{(\sigma^0_{32})^2}{2C^*_{44}} = \frac{(\sigma^0_{31})^2 + (\sigma^0_{32})^2}{2\mu} + \frac{\eta}{2c^2}\int_{-\theta_0}^{\theta_0}\phi(\theta)\int_{C_k}[u_3]t^0_3 dC_k \mathrm{d}\theta, \tag{4}$$

where σ^0_{31} and σ^0_{32} are the anti-plane, far-field stresses, C^*_{44} and C^*_{55} are the two relevant effective moduli of the damaged material, μ is the shear modulus of the undamaged material, c is the average half crack length and $\eta = Mc^2/A$ is the crack density parameter.

If the crack did not exist, the uniform traction due to the far-field stresses, t^0_3, along the line of the crack face is given by

$$t^0_3 = \cos\theta\sigma^0_{32} - \sin\theta\sigma^0_{31}. \tag{5}$$

Next we introduce the GSCM [3] to approximately take into account the interaction between the cracks, as shown in Fig. 1. The inclusion is assumed to have the properties of the undamaged material with known elastic moduli. The surrounding matrix is composed of the effective orthotropic, damaged material with the principal directions along the x_1 and x_2 axes. The half-length of the crack is c, the semi-major and semi-minor axes of the ellipse are a and b, respectively. The crack density parameter η relates average crack length to the area of the ellipse, but in general, this leaves one of the three parameters a, b and c, unspecified. Therefore, as an additional condition, we will require $a^2 - b^2 = c^2$ to be satisfied. This is the same relationship that was used in [3] for convenience, but here it is necessary in order to make analytical solutions possible. The two-phase composite is subjected to uniform anti-plane shearing σ^0_{31} and σ^0_{32} at infinity.

By using the complex variable method [12,13], the crack opening displacement for the elliptical domain, depicted in Fig. 1, $[u_3]$ can be obtained exactly as

$$[u_3] = \frac{4\sqrt{c^2 - x^2}}{\mu[1 + \Gamma - R^{-2}(1 - \Gamma)]}\left[\frac{a\mu^* + bC^*_{55}}{(a+b)\mu^*}\cos\theta\sigma^0_{32} - \frac{a\mu^* + bC^*_{44}}{(a+b)\mu^*}\sin\theta\sigma^0_{31}\right] \quad (|x| \le c) \tag{6}$$