Computers and Fluids 134–135 (2016) 81–89

Contents lists available at ScienceDirect

Computers and Fluids

journal homepage: www.elsevier.com/locate/compfluid

Technical note

Fourth order compact schemes for variable coefficient parabolic problems with mixed derivatives

Shuvam Sen

Department of Mathematical Sciences, Tezpur University, Tezpur 784028, INDIA

ARTICLE INFO

Article history:
Received 22 August 2015
Revised 23 April 2016
Accepted 5 May 2016
Available online 9 May 2016

ABSTRACT

A fourth order compact numerical scheme for variable coefficient parabolic problems with mixed derivatives is outlined. The finite difference scheme, presented here shows good wave resolution property and is stable. Implicit time discretization endows second order temporal accurateness to the scheme. Compact scheme for generalized parabolic 2D convection-diffusion equations is not available in the literature and this work addresses the same. The method is also suitable for computing incompressible flow in arbitrary domains. In this work we have successfully used the method to tackle flows, governed by the incompressible 2D unsteady Navier-Stokes (N-S) equations, in regions beyond rectangular. Further the proposed compact scheme has been found to be proficient in conjunction with numerically generated grids.

1. Introduction

1.1. Problem formulation

Consider parabolic partial differential equation (PDE) of the form

$$\begin{cases} \partial_t\phi(X,t)+A\phi(X,t)=s(X,t), & (X,t)\in\Omega\times(0,T] \\ \phi(X,0)=\phi_0(X), & X\in\Omega \\ b_1(X,t)\phi+b_2(X,t)\partial_{\boldsymbol{n}}\phi=g(X,t), & X\in\partial\Omega,\quad t\in(0,T] \end{cases} \tag{1}$$

for the unknown transport variable $\phi(X, t)$ defined over $\Omega\times(0,T]\subseteq\mathbb{R}^2\times\mathbb{R}$. In this paper we are interested in a compact fourth order numerical discretization of this PDE in a rectangular domain Ω where A is a variable coefficient partial differential operator defined as

$$\begin{aligned}[A\phi](X,t) = &-\alpha_1(X,t)[\partial_{xx}]\phi(X,t)-\beta(X,t)[\partial_{xy}]\phi(X,t)\\ &-\alpha_2(X,t)[\partial_{yy}]\phi(X,t)\\ &+c_1(X,t)[\partial_x]\phi(X,t)+c_2(X,t)[\partial_y]\phi(X,t)\\ &+d(X,t)\phi(X,t)\end{aligned} \tag{2}$$

with $X=(x,y)$. Further the coefficients $\alpha_1(X, t)$, $\alpha_2(X, t)$, $\beta(X, t)$, $c_1(X, t)$, $c_2(X, t)$, $d(X, t)$ and forcing function $s(X, t)$ together with $\phi_0(X)$ and $g(X, t)$ are assumed to be sufficiently smooth. The only additional restriction we have on the Eq. (1) is the positive definiteness of the diffusion matrix. This is equivalent to $\alpha_1(X, t) > 0$, $\alpha_2(X, t) > 0$ and $|\beta(X, t)|^2 \le 4\alpha_1(X, t)\alpha_2(X, t)$ $\forall$ $(X, t)\in\Omega\times(0, T]$. b_1 and b_2 are arbitrary coefficients describing various boundary condition in the boundary normal direction $\boldsymbol{n}$.

The generalized convection-diffusion equations with mixed derivatives, given in Eq. (1), arise in many applications. For example Heston equation, which financial mathematicians use for option pricing in stochastic volatility models [1,2]. We also note the occurrence of such differential equations in mathematical biology [3] and also when coordinate transformations are applied to convection-diffusion equation on non-rectangular domains [4,5]. Such transformations allow us to work on simple rectangular domains or uniform grids independent of the domain of definition of the original problem.

1.2. Prior work

It is well known that higher order compact (HOC) finite difference schemes lead to a system of equations resulting in a coefficient matrix with smaller bandwidth as compared to non-compact schemes. Apart from solving convection-diffusion equation, different compact schemes have been used successfully to solve nonlinear incompressible Navier-Stokes (N-S) equations in all three forms *viz.*, the streamfunction-vorticity [5–11], the primitive variables [12,13] and the biharmonic [14–16] formulations.

Here it is worthwhile to point out that although a plethora of compact schemes [6–11,17–20] have been developed for convection-diffusion equation but few can tackle the generalized one as specified in Eq. (1). Of course a handful of contributions having linkage to the equation can be found in the works of Fournié and Karaa [21], Pandit et al. [5], Karaa [22] and Düring and

E-mail address: shuvam@tezu.ernet.in

http://dx.doi.org/10.1016/j.compfluid.2016.05.002
0045-7930/

Fournié [2]. Fournié and Karaa [21] in 2006 derived a fourth order compact finite difference (FD) scheme for a two-dimensional (2D) elliptic PDE with mixed derivative by considering the PDE itself as an auxiliary relation. But in their work they restricted $\alpha_1 = 1 = \alpha_2$ and also considered β to be a constant with $\beta^2 < 4$. Application of this approach to more general problems such as Eq. (1) is not straightforward and may not be possible. This is accentuated in the work of Pandit et al. [5] in 2007, where the authors tried to derive a compact approximation of parabolic equation. Here the authors were ultimately constrained to work with situations where the mixed derivative is absent. Karaa [22], also in 2007, proposed a fourth order compact FD scheme for solving 2D elliptic and parabolic equations with mixed derivative having variable coefficient by using polynomial approximation, but was again limited by the choice of $\alpha_1 = 1 = \alpha_2$. In 2012, Düring and Fournié [2] used a compact scheme, having fourth order accuracy in space and second order accuracy in time, for option pricing using Heston model. In that manuscript the authors using a variable transformation arrived at a system with $\alpha_1 = \alpha_2$. Note that such transformations are not always certain for a parabolic PDE with variable coefficients.

Although HOC approximation of Eq. (1) is yet to be established, different splitting schemes and their alternating direction implicit (ADI) implementation can be found in the literature. Among them the works of in't Hout and Foulon [1], in't Hout and Welfert [3], Mckee et al. [4] and the references therein deserves special mention. Recently Martinsson [23] has designed a composite spectral collocation scheme for the equation with smooth solutions highlighting importance of generalized parabolic equations. Of late Sen [11] has developed a new family of implicit HOC schemes for unsteady convection-diffusion equation with variable convection coefficient. In this manuscript we generalize this philosophy to propose a HOC formulation for variable coefficient parabolic problem with mixed derivative preserving truncation error of order four in space and two in time. To the best of our knowledge compact schemes for generalized parabolic 2D convection-diffusion equations is not available in the literature and in this paper we intend to address the same. Another aim of the work is to obtain higher order accurate solutions of convection-diffusion equations in domains where it is imperative to use numerically generated grid. All the higher order schemes discussed above fails in conjunction with grids constructed numerically. Our scheme apart from being stable, is efficient in juxtaposition with numerical grids and shows better phase and amplitude error properties. We also augment the newly developed scheme to solve incompressible N-S equations in irregular domains.

2. Fourth order compact schemes for parabolic problem

2.1. Spatial compact discretization

We begin by briefly discussing the development of HOC formulation for the steady form of Eq. (1), which is obtained when α_1, β, α_2, c_1, c_2, d, s and ϕ are independent of t. Under these conditions, Eq. (1) becomes

$$\begin{cases} A\phi(X) = s(X), & X \in \Omega \\ b_1(X)\phi + b_2(X)\partial_{\mathbf{n}}\phi = g(X), & X \in \partial\Omega. \end{cases} \tag{3}$$

For simplicity we assume $\Omega = [a_1, a_2] \times [a_3, a_4]$. In order to obtain a compact spatially fourth order accurate discretization we lay out a grid $a_1 = x_0 < x_1 < \ldots < x_M = a_2$, $a_3 = y_0 < y_1 < \ldots < y_N = a_4$ with $x_i = x_0 + ih$ for $0 \le i \le M$ and $y_j = y_0 + jk$ for $0 \le j \le N$. Consider the following approximations for second order space derivatives appearing in Eq. (3)

$$\partial_{xx}\phi_{i,j} = 2\delta_x^2\phi_{i,j} - \delta_x\phi_{x_{i,j}} + O(h^4), \tag{4}$$

$$\partial_{yy}\phi_{i,j} = 2\delta_y^2\phi_{i,j} - \delta_y\phi_{y_{i,j}} + O(k^4), \tag{5}$$

$$\partial_{xy}\phi_{i,j} = \delta_x\phi_{y_{i,j}} + \delta_y\phi_{x_{i,j}} - \delta_x\delta_y\phi_{i,j} + O(h^2k^2). \tag{6}$$

Here δ_x, δ_y, δ_x^2 and δ_y^2 are usual central difference operators and $\phi_{i,\,j}$ denote the approximate value of $\phi(X_{i,\,j})$ at a typical grid point $X_{i,j} = (x_i, y_j)$. Detailed derivation of the above discretizations can be found in [11]. We thus obtain an $O(h^4, k^4, h^2k^2)$ approximation for Eq. (3) on a nine point stencil as

$$A_{h,k}\phi_{i,j} = s_{i,j} \tag{7}$$

where the discrete operator $A_{h,\,k}$ is defined as

$$\begin{aligned} A_{h,k}\phi_{i,j} = &(-2\alpha_{1_{i,j}}\delta_x^2 - 2\alpha_{2_{i,j}}\delta_y^2 + \beta_{i,j}\delta_x\delta_y + d_{i,j})\phi_{i,j} \\ &+ (\alpha_{1_{i,j}}\delta_x - \beta_{i,j}\delta_y + c_{1_{i,j}})\phi_{x_{i,j}} \\ &+ (\alpha_{2_{i,j}}\delta_y - \beta_{i,j}\delta_x + c_{2_{i,j}})\phi_{y_{i,j}}. \end{aligned} \tag{8}$$

The finite difference operator given above depends on the three grid functions ϕ, ϕ_x and ϕ_y. Compatible fourth order accurate Padé approximations for space derivatives given by

$$\left(I + \frac{h^2}{6}\delta_x^2\right)\phi_{x_{i,j}} = \delta_x\phi_{i,j} \tag{9}$$

and

$$\left(I + \frac{k^2}{6}\delta_y^2\right)\phi_{y_{i,j}} = \delta_y\phi_{i,j} \tag{10}$$

are used to close the system. Note that *vis-a-vis* standard HOC formulation [7], we are not required to approximate the derivatives of the convection coefficients c_1, c_2 and forcing function s. The Eq. (7) can be viewed as a banded system with only nine non zero diagonals; of course drawback of requiring to approximate $\phi_{x_{i,j}}$ and $\phi_{y_{i,j}}$ separately remain.

2.2. Modified wave number analysis

A detailed wave number analysis of the fourth order compact approximation for ϕ_{xx} was carried out by Sen [11]. Here we shall like to examine the characteristic of the newly used fourth order compact approximation for the mixed derivative ϕ_{xy}. Considering the trial function $\tilde{\psi} = e^{\mathbf{I}(\kappa_1 x + \kappa_2 y)}$ $(\mathbf{I} = \sqrt{-1})$ where κ_1 and κ_2 are the wave numbers corresponding to x and y directions respectively, it is easy to see that the fourth order compact discretization for mixed derivative, presented above, has characteristic

$$\lambda_{4OC-M} = -\frac{\sin(\kappa_1 h)\sin(\kappa_2 k)}{hk}\left[\frac{3}{2+\cos(\kappa_1 h)} + \frac{3}{2+\cos(\kappa_2 k)} - 1\right]. \tag{11}$$

To the best of our knowledge no HOC approximation for the mixed derivative is available in literature. To get a clear idea of the dissipation error associated with the proposed discretization we plot the non-dimensional characteristics as a function of $\kappa_1 h$ corresponding to four different values of $\kappa_2 k = 0.5, 1.0, 1.5, 2.0$ in Fig. 1. The figure clearly indicates that the fourth order compact discretization discussed here has superior wave resolution property than other standard discretization procedures *viz.*, the second order accurate central difference approximation (2OC) and fourth order accurate wide stencil approximation (4OW).

2.3. Implicit time discretization

The HOC approach developed for the steady case can be extended directly to the unsteady case by simply replacing s by $s - \partial_t\phi$ in Eq. (3). At grid point $X_{i,\,j}$ at time t, the semi-discrete fourth order scheme for the parabolic equation with variable coefficients will be

$$\partial_t\phi_{i,j}(t) + A_{h,k}\phi_{i,j}(t) = s_{i,j}(t). \tag{12}$$