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# Recent advances on the discontinuous Galerkin method for shallow water equations with topography source terms

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ABSTRACT

We consider in this work the discontinuous Galerkin discretization of the nonlinear shallow water equations on unstructured triangulations. In the recent years, several improvements have been made in the quality of the discontinuous Galerkin approximations for the shallow water equations. In this paper, we first perform a review of the recent methods introduced to ensure the preservation of motionless steady states and robust computations. We then suggest an efficient combination of ingredients that leads to a simple high-order robust and well-balanced scheme, based on the alternative formulation of the equations known as the *pre-balanced* shallow water equations. We show that the preservation of the motionless steady states can be achieved, for an arbitrary order of polynomial expansion. Additionally, the preservation of the positivity of the water height is ensured using the recent method introduced in (Zhang et al., 2012). An extensive set of numerical validations is performed to highlight the efficiency of these approaches. Some accuracy, CPU-time and convergence studies are performed, based on comparisons with analytical solutions or validations against experimental data, for several test cases involving steady states and occurrence of dry areas. Some comparisons with a recent Finite-Volume MUSCL approach are also performed.



## 1. Introduction

The non-linear shallow water equations (SWE in the following) model the dynamic of a free surface shallow layer of homogeneous incompressible fluid. They are used to describe vertically averaged flows in terms of horizontal velocity and depth variation. This set of equations, which can be obtained by asymptotic analysis and depth-averaging of the Navier–Stokes equations [26,34,67], is well-suited for the simulation of geophysical phenomena, such as river and oceanic flows, or even avalanches with suitable source terms. This model is also extensively used in coastal engineering, for the study of nearshore flows involving run-up and run-down on sloping beaches or coastal structures. To allow a proper modelization of such a variety of phenomena, accurate and robust numerical methods have to be considered. Assuming a smooth parameterization of the topography $z : \mathbb{R}^2 \to \mathbb{R}$, the SWE are defined as follows:

$$\mathcal{U}_t + \nabla \cdot H(\mathcal{U}, z) = S(\mathcal{U}, z), \tag{1}$$

with

$$\mathcal{U} = \begin{pmatrix} h \\ q_x \\ q_y \end{pmatrix}, \quad H(\mathcal{U}) = \begin{pmatrix} q_x & q_y \\ uq_x + \frac{1}{2}gh^2 & vq_x \\ uq_y & vq_y + \frac{1}{2}gh^2 \end{pmatrix},$$

$$S(\mathcal{U}, z) = \begin{pmatrix} 0 \\ -ghz_x \\ -ghz_y \end{pmatrix},$$

where $h$ stands for the water height, $\mathbf{u} = (u, v)$ for the horizontal velocity and $\mathbf{q} = (q_x, q_y) = (hu, hv)$ for the horizontal discharge. Denoting by $\Theta$ the convex set of admissible states, defined by:

$$\Theta = \{(h, hu, hv) \in \mathbb{R}^3;\ h \geqslant 0,\ (u, v) \in \mathbb{R}^2\}, \tag{2}$$

we denote $\mathcal{U} : \mathbb{R}^2 \times \mathbb{R}_+ \to \Theta$ the vector of conservative variables, $H : \Theta \to \mathbb{R}^3$ the flux function and $S : \Theta \times \mathbb{R} \to \mathbb{R}^3$ the topography source term.

Nowadays, a large variety of numerical models are able to produce accurate approximations of weak solutions of (1). Finite Volume (FV in the following) methods are known to be very efficient, notably for their low computational cost and their capability

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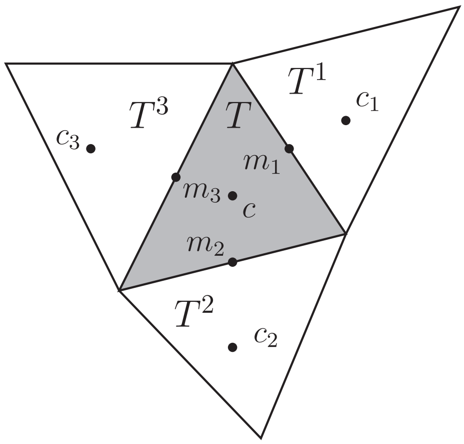


**Fig. 1.** Limitation step: geometric configuration.

in capturing shocks, see for instance [9,27,29,33,36,41,44,70,95] and also some references herein. However, basic FV methods usually offer low accuracy and one generally needs to use some reconstruction methods to offset the low order of convergence and the diffusive losses (see [45,93,75,96] for instance).

Discontinuous Galerkin (DG in the following) methods have raised great interest during the past twenty years. These methods combine the background of the Finite Element methods, FV methods and Riemann solvers, taking into account the physic of the problem. An arbitrary order of accuracy can be obtained with the use of high-order polynomials within elements and they are able to handle complex geometries with the use of unstructured meshes. They are highly parallelizable, and exhibit nice strong stability properties. The reader is referred to the following pioneering works [22,23] for the general backgrounds.

It is only recently that the DG approach has been applied to the SWE and we can find a growing number of studies, including flows with shocks, such as dam-break and oblique hydraulic jumps [60,87,88]. Several approaches involving arbitrary orders methods on unstructured triangulations have been developed for the SWE [31,54], possibly with dynamic p-adaptivity [55], adaptive refinement [81], discretizations of the viscous SWE relying on a Local Discontinuous Galerkin (LDG) treatment of the second order derivatives [1,25], discretizations of the equations on the sphere [6,35,59,69], or even space–time dG methods [2]. The list is of course non-exhaustive.

More recently, several authors have focused on two interesting issues, which are particularly relevant in many applications: the preservation of the motionless steady states, and the preservation of the water height positivity, to properly handle flooding and drying events.

The paper is organized as follows: in the next section, we propose a review of some of the existing methods recently introduced to satisfy these two properties. We also give a review of the main limiting technics introduced to handle discontinuities and prevent the generation of spurious oscillations. In Section 3, we study a new combination of ingredients that lead to an arbitrary order robust and well-balanced nodal discontinuous-Galerkin discretization of the SWE on unstructured meshes, relying on the so called *pre-balanced* SWE [56,84] (PBSW equations in the following) and the recent method introduced in [100,106]. A local limiting process, allowing the possible occurrence of shocks and contact discontinuities, is also described based on [15]. In Section 3.8, we establish the main well-balancing and robustness properties of this combination. Section 4 is devoted to extensive numerical validations in the case of second and third order schemes, including convergence and accuracy analysis, CPU time studies, and comparisons with analytical solutions and experimental data for cases involving steady states preservation and occurrence of dry areas. Some comparisons with the MUSCL FV scheme of [28] are also performed.

## 2. A survey of existing methods

### 2.1. Well-balancing

This property is often referred to as *C-property*, following [5]. Defining the free surface $\eta = h + z$, we say that the motionless equilibrium states are preserved by a given numerical scheme if the following property holds for all $n \in \mathbb{N}$:

$$\left( \begin{cases} \eta_{\mathfrak{h}}^n \equiv \eta^e \\ \mathbf{q}_{\mathfrak{h}}^n \equiv 0 \end{cases} \right) \Rightarrow \left( \begin{cases} \eta_{\mathfrak{h}}^{n+1} \equiv \eta^e \\ \mathbf{q}_{\mathfrak{h}}^{n+1} \equiv 0 \end{cases} \right), \tag{3}$$

where $\eta^e$ is a constant, and $\mathbf{w}_{\mathfrak{h}} = (\eta_{\mathfrak{h}}, \mathbf{q}_{\mathfrak{h}})$ is the discrete solution produced by the numerical scheme.

Nowadays, a large number of FV approaches are able to offer such a property, see among them [3,5,28,33,37,40,46,48,56–58,62,64,86,108] for first and second order accuracy well-balanced FVM and [16,18,71,83,97,99] for some higher order schemes.

The development of well-balanced DG schemes for the SWE is recent, and there is very few existing works, especially when considering the 2D case on unstructured grids. In [80], general space and space–time DG formulations are introduced for hyperbolic nonconservative partial differential equations, and applications are performed for the one-dimensional SWE with topography, regarding the topography as an additional variable in the spirit of [40]. The resulting space method is shown to preserve the *C-property*.

A well-balanced method is developed for second order accuracy DG schemes in [49–51], for the 1D and 2D case on rectangular meshes, using the PBSW equations and borrowing some ideas coming from the *hydrostatic reconstruction* [3,62].

In [30], the well-balancing is ensured for polynomial expansions of arbitrary orders and on unstructured meshes, using the ideas of the *hydrostatic reconstruction*. Non-negative reconstructions of the water height are introduced element-wise, for each edge, together with an additional flux modification term directly accounted for in the weak formulation.

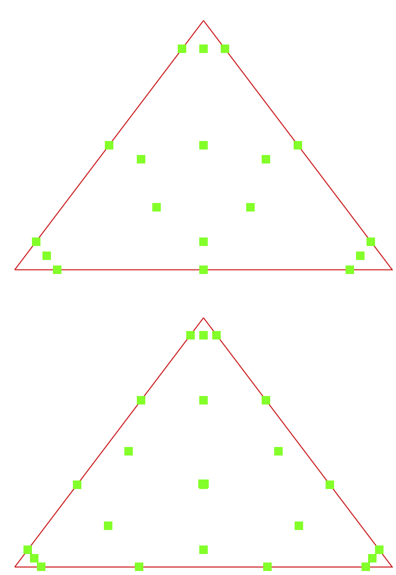

**Fig. 2.** Nodes locations for the Zhang and Shu quadrature – $\mathbb{P}^1$ and $\mathbb{P}^2$ cases.