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# A conservative, skew-symmetric finite difference scheme for the compressible Navier–Stokes equations

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ABSTRACT

We present a fully conservative, skew-symmetric finite difference scheme on transformed grids. The skew-symmetry preserves the kinetic energy by first principles, simultaneously avoiding a central instability mechanism and numerical damping. In contrast to other skew-symmetric schemes no special averaging procedures are needed. Instead, the scheme builds purely on point-wise operations and derivatives. Any explicit and central derivative can be used, permitting high order and great freedom to optimize the scheme otherwise. This also allows the simple adaption of existing finite difference schemes to improve their stability and damping properties. Numerical examples covering acoustic phenomena and shocks are presented.



## 1. Introduction

Flows featuring shocks and acoustics are common in engineering applications. Examples are the sound generation of supersonic jets, noise generation in transonic flight or shock buffeting on an airfoil, where one possible mechanism is the interaction of sound waves from the trailing edge with the standing shock on the airfoil.

Simulations of such configurations are numerically challenging as the correct and stable simulation of shocks and the faithful calculation of acoustics have different, and in standard approaches, contradictory requirements. Shocks, on the one hand, are described by the Rankine–Hugoniot conditions, which build directly on the conservation laws. Conservation of mass, momentum and total energy is a prerequisite to guarantee a correct shock treatment. Such conserving schemes are usually finite volume (FV) schemes.

Acoustics, on the other hand, are propagating disturbances. They experience very low damping and travel long distances without noticeable energy loss in practice. When the amplitude is small the dispersion is also very small. To preserve these properties in a numerical simulation, schemes with very low (numerical) damping and low dispersion are needed, so that high-order, dispersion optimized finite difference (FD) schemes are mostly used.

In contrasto finite volume, most finite difference schemes are only approximately conserving, becoming worse, where flow variables are rapidly changing. Thus, in shocks, where conservation is most desired they become unreliable. Indeed, it is well known that FD schemes can totally fail to describe shocks. On the other hand FV usually utilize upwind schemes for stability reasons, leading to high artificial damping. Even the optimized schemes hardly reach the quality of FD schemes for acoustics simulations. Thus a conservative scheme with low damping is desirable.

To understand the key point of a low damping scheme consider the momentum equation for the $\alpha$th velocity component, $(\alpha, \beta \in 1,2,3)$, $\mathbf{u} = (u_1, u_2, u_3)^t$.

$$\partial_t \varrho u_\alpha + \partial_{x_\beta}\left(\varrho u_\beta u_\alpha\right) + \partial_{x_\alpha} p = \partial_{x_\beta} \tau_{\alpha\beta}.$$

As usual $p$ is the pressure and $\tau_{\alpha\beta} = \mu(\partial_{x_\alpha} u_\beta + \partial_{x_\beta} u_\alpha) + (\mu_d - \mu 2/3)\delta_{\alpha\beta}\partial_{x_\gamma} u_\gamma$ the friction. Summing convention is assumed. From this the equation for the kinetic energy $E_{kin} = \varrho u_\alpha u_\alpha/2$ is derived with the help of the mass conservation[1] as

$$\partial_t \varrho u_\alpha u_\alpha/2 + \partial_{x_\beta}\left(\varrho u_\beta u_\alpha u_\alpha/2\right) = -u_\alpha \partial_{x_\alpha} p + u_\alpha \partial_{x_\beta} \tau_{\alpha\beta}.$$

Only pressure work and friction changes the kinetic energy. The transport of the kinetic energy is in contrast strictly conservative. This physical property is easily destroyed in numerical schemes. Up-winding destroys the conservativity of the transport by introducing artificial damping; but even a central derivative usually does not exactly preserve the kinetic energy. Thus a transport term, which conserves the kinetic energy is the key to an undamped simulation. It should be pointed out, that an artificial damping does not destroy the conservation of the *total energy* in FV, as by construction the lost kinetic energy is balanced by an increased internal energy.

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[1] This derivation will be shown in more detail further down.

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But the kinetic energy of, say, sound waves is irreversible transformed to internal energy, thus to heat.

The correct treatment of the kinetic energy is the focus of skew-symmetric schemes. The skew symmetric schemes were first introduced by Feiereisen et al. [1] and Tadmor [2]. While Feiereisen was focusing on numerical simulations Tadmor concentrated on analytical aspects. The discretization of the non-linear transport term is chosen, so that the implied term for the equation of the kinetic energy is strictly conservative. This is achieved by formulating the transport term as a skew-symmetric operator, which implies the conservation of kinetic energy by first principles: the change of kinetic energy is calculated from a quadratic form of this transport term, which is zero, as all quadratic forms of skew-symmetric matrices are zero. The main challenge then is to preserve the conservation of the mass, momentum and total energy; especially the conservation of momentum is far from obvious and might be violated. To this end it has to be possible to rewrite the discrete skew-symmetric operator into a term with the telescoping sum property, typically to a discrete form of the divergence.

The classical ansatz to ensure this telescoping sum property of the skew-symmetric operator is to use skillful averaging of different variables. This was successfully applied by Morinishi in [3] and became standard in this area. This procedure is by no means the only way to obtain skew-symmetry and the telescoping sum property; we find that no such averaging is necessary. A straightforward and consistent discretization is sufficient. This not only leads to simpler expressions and to simple proofs of all claimed properties, but it also allows to rewrite existing FD codes with minimal effort to a skew-symmetric form, by simply changing the spatial discretization, yielding good stability properties and low numerical damping. For strictly conservative schemes the time-stepping has to be changed accordingly; a dedicated paper about different time integration is submitted, [4]. This paper also shows how a conserved norm can be constructed, which implies stability.

Two points related with stability should be considered. First, stability does not imply that the solution is physical. Indeed strong and unwanted oscillations do appear if the chosen dissipation is too small for a given discontinuity or for strong grid stretching. Here, numerical damping has to be added. Still we prefer adding dissipation explicitly over creating dissipation by the scheme itself, as this allows to control the amount of dissipation and tailor it for a given situation. This also allows to explicitly define the damping, which can be helpful for turbulence modeling. Secondly, one can only hope for stability as long as one can solve the implicit time stepping. This is in our experience a minor problem for moderate time steps even for solutions which are unphysical due to strong oscillations.

In contrast to the usual averaging procedure, the structure derived in this paper builds on matrices and makes little reference on the details of the stencil. Instead, abstract properties, namely the skew-symmetry and the telescoping sum property are assumed, which are fulfilled by basically all central and in computational space equidistant derivatives. These properties permit one to choose a derivative to one's needs, be it any high order or wave-number optimized derivative. High order is obtained by using a standard (explicit) high order derivative. We make use of this freedom by choosing a derivative with the so called summation by parts property, [5], which allows clean and flexible boundary conditions without using ghost points.

For practical calculation curvilinear grids are essential. It is understood, [6], that a grid transformation to a computational space is a suitable way to preserve the correct structure on curvilinear grids. To our knowledge the first to obtain this for compressible flows was Kok, [7]; the authors presented a similar procedure for our FD scheme in [8,9]. During submission we became aware of [10], which builds on similar ideas and extends it even to moving grids. Also [11] and more recently, [12] investigated skew symmetric schemes on curvilinear grids.

A third way to derive skew symmetric-schemes is the Galerkin ansatz. Products in the function space can be approximated leading to a numerically effective scheme, see e.g. Gassner [13]. The approximation leads a skew-symmetric scheme for the Burgers equation with similar point-wise products and derivatives as derived in this paper; the summation by parts property which is a choice in this paper occurs naturally in the work by Gassner.

This paper is organized as follows: first we introduce the rewriting and our way of spatial discretization of the Navier–Stokes equations in one dimension in Section 2. Then we derive the three-dimensional equations, introduce curvilinear grids and discretize in Section 3. A time discretization which generalizes the scheme of Morinishi [14] and Subbareddy et al. [15] is derived in Section 4. Boundaries are discussed in Section 5. We close with some numerical examples in Section 6. In the appendix we first compare our scheme with the standard, averaging approach. Finally we show by construction, that the FD scheme implies local and consistent fluxes.

## 2. The Navier–Stokes equations in one dimension

The Navier–Stokes equations in one dimension are given by

$$\partial_t \varrho + \partial_x(\varrho u) = 0 \tag{1}$$

$$\partial_t(\varrho u) + \partial_x(\varrho u^2) + \partial_x p = \partial_x \tau \tag{2}$$

$$\partial_t(\varrho e + \varrho u^2/2) + \partial_x\big(\varrho u(e + p/\varrho + u^2/2)\big) = \partial_x u\tau - \partial_x \phi. \tag{3}$$

They describe mass, momentum and energy conservation. In the following the internal energy of the ideal gas $e = (p/\varrho)/(\gamma - 1)$ is assumed with the constant adiabatic index $\gamma$. Perfect gas is assumed in many simulations. However, we emphasize that the following steps concern only the kinetic energy, so that any other equation of state could be used. The friction in one dimension is $\tau = \mu\partial_x u$ with $\mu$ the viscosity and $\phi = -\lambda\partial_x T$ is the heat flux, with the heat conductivity $\lambda$.

The transport term in the momentum Eq. (2) given in divergence form, can be rewritten to convective form with help of the mass conservation as

$$\partial_t(\varrho u) + \partial_x(\varrho u^2) = \varrho\partial_t u + \varrho u\partial_x u.$$

Adding the divergence and convective form of the momentum equation we obtain the skew-symmetric form. The Navier–Stokes equations become

$$\partial_t \varrho + \partial_x(\varrho u) = 0 \tag{4}$$

$$\frac{1}{2}(\partial_t\varrho\,\cdot + \varrho\partial_t\cdot)u + \frac{1}{2}(\partial_x u\varrho\,\cdot + u\varrho\partial_x\cdot)u + \partial_x p = \partial_x\tau \tag{5}$$

$$\frac{1}{\gamma - 1}\partial_t p + \frac{\gamma}{\gamma - 1}\partial_x(up) - u\partial_x p = -u\partial_x\tau + \partial_x u\tau - \partial_x\phi, \tag{6}$$

where it is understood that the space and time derivatives in the first two terms of (5) act also on $u$ right of the parentheses. For clarity this is explicitly marked by a dot "·". The kinetic energy was split off from the energy equation with the help of the momentum equation by use of the product rule

$$\partial_t(\varrho u^2/2) + \partial_x\big(\varrho u(u^2/2)\big) = \frac{1}{2}u(\partial_t\varrho\,\cdot + \varrho\partial_t\cdot)u + \frac{1}{2}u(\partial_x(\varrho u)\cdot + (\varrho u)\partial_x\cdot)u = -u\partial_x p + u\partial_x\tau, \tag{7}$$

leaving just the pressure work $-u\partial_x p$ and a friction term. In the second line the skew-symmetric form of the momentum transport term appears, which underlines its close connection to the kinetic energy.