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Conservative form and spurious solutions in moving boundary elements methods

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ABSTRACT

In this study the Fredholm integral equations of the first and the second kind have been examined using two numerical approaches: "adding and subtracting of the singularity" and "piecewise linear panels" methods. While, the solution for the Fredholm integral equation of the first kind (kinematic condition-no flux) shows good agreement with the analytical solution, the Fredholm integral equation of the second kind generates spurious results only in the case of the piecewise linear panels method. It was found that the spurious solutions are a direct result of the low order of numerical scheme accuracy in the piecewise linear panels method. In order to correct the structure of the difference scheme in the piecewise linear panels method, the numerical error was redistributed to preserve the conservative form of the circulation at the difference scheme level. The obtained solution in the conservative form does not exhibit any spurious results. This has direct consequences on a moving boundary such as a free interface, where it is shown that the non conservative form of the "piecewise linear panels" difference scheme is exhibiting spurious results, while the conservative form has a good agreement with the analytical solution.

1. Introduction

The boundary integral method is well known as an efficient approach for solving potential flow problems. The main advantage of this technique is that the flow-field solution is reduced to discretizing the internal boundaries (e.g. hydrodynamic bodies or cavity interfaces) as opposed to the entire fluid domain, significantly reducing necessary computer resources such as memory capacity and computational time. Integral techniques based on the solution of Green's function have become an important engineering tool. For such methods the hydrodynamic body surface is divided into small panels and a combination of vortex or source distributions are distributed on each of the panels. In the lowest order approximation, such as that of the vortex lattice method, a concentrated vortex is placed on each linear panel [10]. An improvement to this approach is that of Hess and Smith [9], where the flow-field was represented by a source distribution plus a circulation term. The main advantage of this approach is the integration technique, in which the source distribution was integrated along each panel. However, the most advanced methods today are based on integrating higher order vortex sheet or source distributions over curved panels [14,15]. The end result of all these methods is the generation of a full matrix of influence in which the no flux condition or the kinematic boundary condition is satisfied at the collocation points located on each panel. Therefore, the vortex sheet representation (note that the dipole representation in Green's function can be replaced by a vortex sheet) with the no flux condition is essentially reduced to the well-known Fredholm integral equation of the first kind [5,13].

An alternative approach for solving the integral equations that is less widely used in the engineering field is based on solving the Fredholm integral equations of the first and second kind [2,8,11]. The basic concept in this approach is to eliminate the singularity in the singular Cauchy integral (which is no more than Green's formulation, but in the complex plane) in a way that the new integrand in the integral becomes a smooth function. This is achieved by adding and subtracting the term of the singularity. The resulting equation in combination with a period boundary condition and smooth integrand results in a methodology that is highly accurate compared to the high order panels (We will elaborate upon this matter in Section 3.2). Another approach that deserve attention is based on the Euler Maclaurin expansions of the trapezoidal rule in which high-accuracy numerical quadrature methods for integrals with a singular periodic integrand has been proposed by Sidi and Israeli [17]. In our study we will treat the same singular integral in a different manner.

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Baker et al. [2] introduced a different approach by using a dipole strength distribution on the surface, which is spectrally accurate and can be solved quickly and efficiently through iterative methods. These methods have been applied successfully in 2D motion and they have been extended to 3D motion [21]. The vortex blob methods are a different approach to deal with moving boundaries and they are based on replacing the kernel of the Fredholm integral equation either of the first kind or the second kind by a regularized kernel and then evaluating the integral numerically [1]. Vortex blob approaches provide a regularization for the motion of vortex sheets [3].

There is a large body of work on conformal maps through the use of Cauchy's integral and its numerical approximation. In this regard, [12] describes an integral equation method for computing the conformal mapping of multiply connected regions onto an annulus with concentric circular slits. While, Crowdy and Marshall [6] has constructed an explicit analytical formula for the conformal mappings from the canonical class of multiply connected circular domains to canonical classes of multiply connected slit domains.

For problems dealing with a moving boundary, such as a free interface, one must determine how well each method identified above is capable of solving the moving boundary. However, in this work we will restrict ourselves to the piecewise linear panels method, and for an accurate scheme we will adopt the Fredholm integral approach of adding and subtracting the singularity (see e.g., [2,8,11]). Preliminary work on the 2-dimensional problem, has shown that the piecewise linear panels method can exhibit a spurious solution. Also, Ref. [16] has reported on spurious results using the same equations and same numerical approach. On the contrary, the adding and subtracting the singularity approach does not show any aspects of irregularity in the solution and does not exhibit a spurious solution. The fact that a spurious solution can be obtained when employing a piecewise linear panel method, has motivated this study. The goal is to identify the source and mechanism of the abnormality, and if possible, to identify ways to avoid and correct this behavior. It was found in this research that the piecewise linear panels method is essentially carrying a low order truncation error compared to the adding and subtracting the singularity technique (and probably to higher order accurate panel schemes). Therefore, the matrix of influence of such a low order accurate scheme must be revised in a way that the discretized scheme preserves the conservative form of the governing equation. This was accomplished by redistributing the truncation error in a way that preserves the conservative form in the same sense as in the continuous equation.

Moreover, most of the methods involving 3-dimensional moving boundaries are based upon piecewise linear panels methods. Therefore, it is important to determine if such abnormal behavior exists for the 3-dimensional case and to what order of numerical accuracy the panel method should be to avoid spurious solutions. It would also be important to determine, whether a similar approach to that presented here can be implemented in the case of 3-dimensions in order to prevent spurious solutions.

2. Fredholm integral equations and Cauchy integral

The Cauchy integral of any analytical function on a simply connected domain can be written as,

$$\oint \frac{\frac{dw}{d\xi}}{\xi - z} d\xi = 0,$$

where $w = \phi + i\psi$ is the complex potential function, $z = x + iy$ is the coordinates of a given point at (x, y) in the complex plane, and ξ is the integration variable. Note that ϕ and ψ are the velocity potential and stream function, respectively, and $dw/d\xi = u(\xi) - iv(\xi)$ is the velocity in the complex form. The above equation is true only when the pole remains outside of the domain of integration. For a multiply connected domain, the domain is divided into the form of C_1, C_2, C_3, as shown in Fig. 1. Using this approach it can be shown that,

$$-2\pi i \frac{dw}{dz} + 2U_\infty \pi i e^{-i\alpha} + \oint \frac{\frac{dw}{d\xi}}{\xi - z} d\xi = 0, \tag{1}$$

where U_∞ is the free stream velocity and α is the angle of attack. The first term in Eq. (1) is the integration result along C_1, the second term is the integration along the far field C_3 (i.e., as $|z| \to \infty$), and the last term is the Cauchy integral along the foil. Along the interface the Cauchy integral reduces to,

$$-\pi i \frac{dw}{dz} + 2U_\infty \pi i e^{-i\alpha} + \oint \frac{\frac{dw}{d\xi}}{\xi - z} d\xi = 0. \tag{2}$$

Eq. (2) is obtained when z lies on the body interface or C_1 lies on C_2, therefore half a circle is being taken rather then a full circle for C_1. The same expression can be obtained in a different manner by assuming that z is a small distance ϵ from the foil interface and carrying out the asymptotic expansion of Eq. (1). It can be shown that in the limit case when $\epsilon \to 0$ Eq. (1) reduces to Eq. (2).

Since $\frac{dw}{dz} = u - iv = (U_s - iV_n)e^{-i\theta}$, where u, v, U_s and V_n are defined in Fig. 2, Eq. (2) can be rewritten as,

$$U_s - iV_n = 2U_\infty e^{i(\theta - \alpha)} + \frac{e^{i\theta}}{\pi i} \oint \frac{U_s - iV_n}{\xi - z} ds, \tag{3}$$

where s is also defined in Fig. 2, and the inclination angle, θ, is the slope of the collocation points along the body interface. Note that in Eq. (2) $\xi = \xi(s)$ while z is the collocation point along the rigid body interface.

The integral equation of the inner flow-field inside the body for any point z outside the body/foil can be written as,

$$\frac{1}{2\pi i} \oint \frac{U_s^- - iV_n^-}{\xi - z} ds = 0. \tag{4}$$

The uniqueness of the elliptic equation in internal flow inside the body requires that, $Q = \oint V_n^- ds = 0$, where Q is the mass flux across the foil interface. That is, if the discharge Q is not zero, there must be a source term in the internal flow-field (see Fig. 1). Therefore, the solution in terms of the velocity potential can be

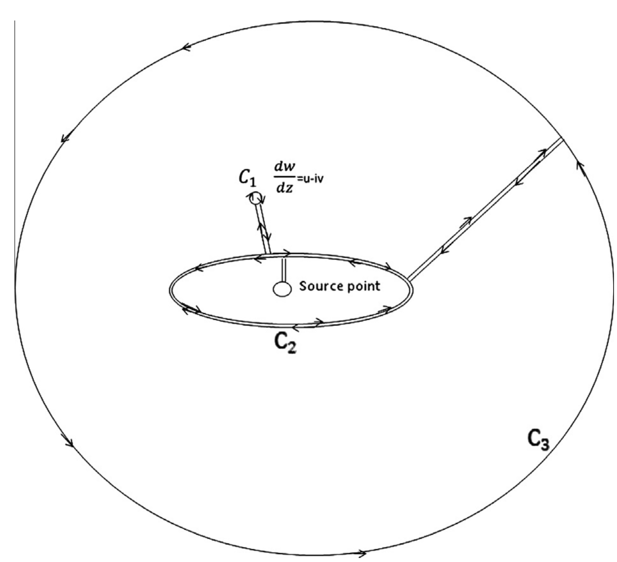

Fig. 1. Schematic description of the complex domain of integration.