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Some aspects relating to the accuracy of high-order panel methods

A. Dagan *, T. Jeans, R.H. Cooper [1]

Department of Mechanical Engineering, University of New Brunswick Fredericton, NB E3B 5A3, Canada

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ABSTRACT

In this study we have examined two numerical algorithms, based upon the high-order panels approach, to verify and demonstrate that a spurious solution is a direct result of a low-order scheme accuracy that violates the conservation of circulation. The first algorithm is based upon the parabolic Lagrange interpolation and the second one is based on the same parabolic Lagrange interpolation for the vortex sheet strength where the geometry of the body/foil is evaluated using a periodic cubic spline, mainly because the denominator in the integrand of the Cauchy algorithm is the main source of the numerical error in the scheme. Good agreement has been found between the computational and analytical results and a spurious free solution has been obtained for the high-order schemes, except for a spurious-like solution in the case of an under resolved problem.

1. Introduction

The boundary integral methods are well known as efficient approaches for solving potential flow problems. The main advantages of such techniques arise from the fact that the flow-field solution is reduced from a volume approach to a surface method. This requires less computational resources such as memory and computational time. As powerful engineering methods, these types of integral techniques which are based on the Green's function solution have become important engineering tools. The common ground for these methods is the division of the aerodynamic surface into small panels and the placing of a vortex distribution on each one of the panels. Among the variety of vortex methods is the vortex lattice method (VLM) in which a concentrated vortex is placed in each panel [9]. An improvement to the VLM is the Smith and Hess approach [6] where the flow field is represented by a source distribution and a circulation term. The main advantage in the latter approach is the technique of integrating the source distribution over the panels. However, the most advanced methods used today are based on the integration technique of a high-order vortex sheet distribution or source distribution over curved panels. The end result of all of these methods is the generation of a full matrix of influence in which the no-flux condition or the kinematic boundary condition is satisfied at each collocation point of each panel. Therefore, the vortex sheet representation (the dipole representation in the Green's function can be replaced by a vortex sheet distribution) with a no-flux condition is essentially reduced to the well-known Fredholm integral equation of the first kind [3] [12].

There is a rich literature on the discretization of singular integral operators of the type considered here. Various Galerkin [1,7] and Nystrom schemes have been developed [5,10]. high-order algebraic convergence can be achieved via corrected trapezoid rules and generalized Gaussian quadratures where exponential convergence can be obtained [13]. Moreover, there is an extensive framework for understanding the convergence of Nystrom and Galkerin type approximations of singular integral operators. Among the variety of quadrature methods an approach to the same integral equations was employed by [5,11,2]. The basic concept in their approach (which is a basic standard in quadrature methods) was to eliminate the singularity in the singular Cauchy integral (which is no more than the Green's formulation in the complex plane) in such a way that the new integrand in the integral becomes a smooth function. This was done by adding and subtracting the term of the singularity. The resulting equation combined with the periodical boundary condition has enabled the approach to be highly accurate compared to the high-order panel method. Unfortunately, to our knowledge there is no extension of this approach to a three-dimensional case although Ref. [10] has implemented a similar approach in 3-D.

It has been demonstrated by Ref. [4] that first-order schemes, such as "piecewise linear panel" methods, are carrying low-order of truncation errors which generate a spurious result. Also, Ref. [15] has reported on spurious results using the same equations and numerical approach while [16] claims that the matrix conditioning of the resulting equation is the main cause of the inaccurate

* Corresponding author. Tel.: +1 506 453 5018.
E-mail addresses: ariedag@gmail.com (A. Dagan), tjeans@unb.ca (T. Jeans), cooper@unb.ca (R.H. Cooper).
[1] Faculty of Computer Science, UNB.

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results. On the other hand, Ref. [8] has dealt with the same equation using a high-order panel technique in which the cubic spline approach was incorporated for the vortex strength distribution; he also reported substantial difficulties in solving this equation. Contrary to low-order accuracy panel methods, high-order accurate schemes, such as the approach of "adding and subtracting of the singularity" (see [4,5,11,2]), do not exhibit abnormalities, such as those found using "piecewise linear panel" methods, and do not exhibit a spurious solution. The fact that a spurious solution can be obtained when employing a low-order scheme, such as "piecewise linear panel" methods, has motivated this study, since it is highly related to the appearance of spurious results in the case of moving boundaries [4]. The goal is to identify the source and mechanism of the abnormality, and if possible to find ways to avoid and correct this behavior. Therefore, in this work we will check the sensitivity of the high-order accurate schemes that are based upon high-order panel methods in order to verify and to demonstrate that the spurious solution is the outcome from a low-order accurate schemes. For this purpose, we have employed two methods that are based on the higher order panels. In the first approach the vortex sheet was represented by a second-order polynomial interpolation (forward Lagrangian interpolation) to represent the vortex sheet strength and the foil/body geometry. Since the denominator in the Cauchy integrand is the main source of numerical error, this approach was modified to represent the foil geometry with a periodical cubic spline. It was found that, in general, the higher order panels do not exhibit a spurious solution. However, for under-resolved problems, it seems that the higher order panel method based on the Lagrangian interpolation exhibits a spurious behavior. This can be corrected by applying the same procedure that was suggested by Ref. [4]. It is clear from this research that 2-D low-order schemes generate a spurious solution and therefore only high-order schemes can be taken under consideration which is a computationally heavy burden in terms of computer operations and programming effort. These points have to be further investigated in 3-D to verify that the same conclusions are also true in the 3-D cases, and to see which order of numerical accuracy the panel method should be to avoid spurious solutions. This point has high relevance in those cases where the accuracy of high-order schemes deteriorate, such as in the cases of under-resolved problems or inadequate mesh resolution where in such events high-order schemes can be turned into low-order schemes. It would be rather interesting to see whether or not a similar conservative approach in the fashion of Ref. [4] can be implemented in the case of three dimensions in order to prevent spurious solutions.

2. The governing equations – the Fredholm integrals

In this section we will briefly introduce the governing equations of the flow field. For more clarification see Ref. [4].

On a simply connected domain that does not contain the point z, any analytical function $\frac{dW}{dz}$ can be written in the form of the Cauchy integral as,

$$\oint_{\Omega} \frac{\frac{dW}{d\xi}}{\xi - z} d\xi = 0$$

where $W = \phi + i\psi$ is the complex potential function and the variables ϕ and ψ are the velocity potential and stream function respectively. The term $dW/d\xi = u(\xi) - iv(\xi)$ is the conjugate velocity in the complex form, where the velocity components (u, v) are along the (x, y) direction respectively (see also Fig. 1). The integration variable ξ is along the boundaries, where $z = x + iy$ is the coordinates of a given point z at (x, y) in the complex plane, while. Ω denotes path of the integration along the boundary of a simply connected domain. The above equation can be written in the following form:

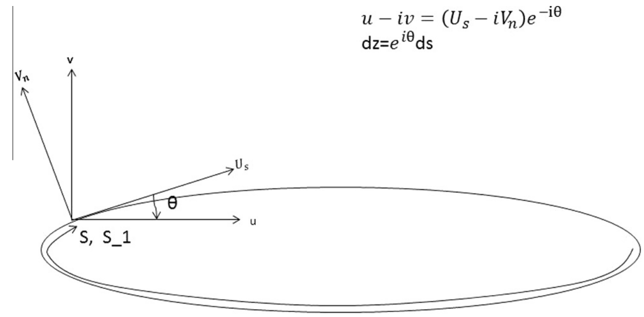

Fig. 1. Schematic description of the body/foil.

$$-(1 + \kappa(|z|))\pi i \frac{dw}{dz} + 2U_\infty \pi i e^{-i\alpha} + \oint_S \frac{\frac{dw}{d\xi}}{\xi - z} d\xi = 0 \quad (1)$$

The term $\kappa(|z|)$ is the Heaviside step function in which $\kappa(|z|) = 1$ everywhere and $\kappa(|z|) = 0$ when z lies on the body surface, while U_∞ is the background velocity and α is the angle of attack. The first term in Eq. (1) is the integration result along any arbitrary point in the fluid domain, while the second term represents the contour integration along the far field (i.e., as $|z| \to \infty$). The last term is the Cauchy integral along the foil/body. The complex conjugate velocity can be expressed as: $\frac{dw}{dz} = u - iv = (U_s - iV_n)e^{-i\theta}$, where u, v, U_s and V_n are defined in Fig. 1, therefore, Eq. (1) along the body/foil surface can be further reduced,

$$U_s - iV_n = 2U_\infty e^{i(\theta - \alpha)} + \frac{e^{i\theta}}{\pi i} \oint_{S^*} \frac{U_s - iV_n}{\xi - z} ds \quad (2)$$

Note that $\xi = \xi(s)$ in Eq. (1) while z is the collocation point along the foil/body interface. The variable θ, is the surface slope at the collocation points, where s is the arc length along the body/foil surface, defined in Fig. 1, while S represents closed integration along the body surface and S^* denotes the Cauchy principal value over the closed surface. In a fashion similar to Ref. [4], the inner solution of the flow-field inside the body can be written as,

$$\frac{1}{2\pi i} \oint_S \frac{U_s^- - iV_n^-}{\xi - z} ds = 0 \quad (3)$$

In the present work we will continue the discussion of the high-order panel methods in the case of $Q = \oint_S V_n^- ds = 0$, where Q is the mass flux across the foil interface. However, some concern has been addressed in the case where $Q = \oint_S V_n^- ds \neq 0$, (see Ref.[4] for more clarification).

Upon imposing a continuous $V_n = V_n^-$ across the body interface the following Fredholm equation is obtained,

$$-2U_\infty sin(\theta - \alpha) + \Re\left\{\frac{e^{i\theta}}{\pi} \oint_{S^*} \frac{\gamma}{\xi - z} ds\right\} = V_n \quad (4)$$

and

$$2U_\infty cos(\theta - \alpha) + \Im m\left\{\frac{e^{i\theta}}{\pi} \oint_{S^*} \frac{\gamma}{\xi - z} ds\right\} = U_s \quad (5)$$

The new auxiliary variable $\gamma = U_s - U_s^-$ is the strength of the vortex sheet distribution along the rigid body surface. However, when zero mass flux ($V_n = 0$) is under consideration, the strength of the vortex sheet distributions reduces to $\gamma = U_s$, where the inner velocities U_s^-, V_n^- are zero. In such a case, Eq. (4) is defined as Fredholm integral of the first kind, and Eq. (5) is reduced to the Fredholm integral of the second kind. For the case of zero flux across the solid boundary Eqs. (4) and (5) are identical to (2). In the case of a moving boundary or jet injection the normal velocity is not zero ($V_n \neq 0$). Consequently, the flow-field is represented by a vortex distribution of strength $\gamma(s)$ that does not necessarily