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Analytical solution of plane Couette flow in the transition regime and comparison with Direct Simulation Monte Carlo data

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ABSTRACT

This paper provides analytical solution for the steady-state Couette flow problem in the transition flow regime, while capturing the non-linear Knudsen layer near the walls. Slope at the center obtained from Direct Simulation Monte Carlo (DSMC) data and inherent symmetry in the problem have been utilized for obtaining the solution. A detailed study of the solutions obtained from the linearized super-Burnett, augmented Burnett and R13 equations is presented. The analytical results are compared against DSMC data; good agreement between them is shown till $Kn = 10$. These are among the first set of analytical solutions in the transition regime. The results indicate that the solution tends to become linear as the Knudsen number increases. The results have allowed formulation of a slip relationship, which can potentially yield more accurate slip velocity than Maxwell's slip model in the transition regime. Our analysis suggests that the Knudsen number envelope over which the R13 and higher-order continuum equations can be employed is substantially extendable.

1. Introduction

Knudsen number (Kn), defined as the mean free path of the gas (λ) divided by the characteristic length scale, can be used to demarcate flow into four regimes. These regimes are: continuum ($Kn < 10^{-3}$), slip ($10^{-3} < Kn < 10^{-1}$), transition ($10^{-1} < Kn < 10$) and free-molecular ($Kn > 10$). The conventional Navier–Stokes–Fourier equations are applicable in the continuum regime; solutions of these equations for a large number of different cases are available in standard textbooks. Similarly, owing to only a few molecules in the free-molecular regime, this last regime can also be described reasonably well. The challenge lies in describing the flow in the intermediate (slip and transition) regimes. Whereas the application of the slip boundary condition along with the Navier–Stokes equations has been found to give reasonably accurate results in the slip regime [1–4], analytical solution for flow in the transition regime appears much more difficult to obtain. This is because the Navier–Stokes equations are no longer applicable in this regime and both inter-molecular and molecule-wall collisions have to be considered. The aim then is to apply higher-order constitutive equations (such as Burnett, super-Burnett, Grad's-moment or R13 equations) and obtain their analytical solution. These equations besides being non-linear require additional boundary conditions; obtaining analytical solution of these equations therefore becomes particularly challenging. The present work discusses analytical solution of these higher-order constitutive equations in the transition regime for the plane-Couette flow problem.

The conventional fluid dynamics equations such as Navier–Stokes, which are first-order equations in terms of Knudsen number, are applicable for small values of the Knudsen number ($Kn \ll 1$). At high values of Knudsen number, these equations show deviation close to a surface and fail to capture the non-linear stress/strain-rate relationship resulting in the Knudsen layer near the surface. Although fictitious slip boundary condition captures the flow outside the Knudsen layer fairly well, the match with DSMC data within the Knudsen layer is rather poor [5]. The next logical approach is to use higher-order continuum equations. The Burnett and super-Burnett equations are obtained by a Chapman–Enskog expansion of the Boltzmann equation, with Knudsen number as the parameter [6–8]; these equations are second- and third-order accurate in terms of Knudsen number, respectively. The moment equations are obtained by taking moments of the Boltzmann equation with additional equations obtained from the distribution function for closure [9–11]. The moment equations retain all higher-order terms of Knudsen number. These higher-order equations have shown merit by correctly predicting one-dimensional shock wave thickness in rarefied hypersonic flows and other aerospace related problems; see Salomons and Mareschal [12] for the merit of Burnett equations in shock waves

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and Wuest [13] where comparison of solutions from various equations has been presented.

However, the integrity of the Burnett and super-Burnett equations has been questioned by some researchers. For example, the Burnett equations suggest a linear velocity profile for the steady-state Couette flow problem (similar to the Navier–Stokes equations), thus failing to capture the non-linearity in the solution. Similarly, the nature of the solution obtained by solving the super-Burnett equations for the Couette flow problem shows that both velocity and temperature are oscillatory in space; which appears unphysical at first sight. It was shown by Struchtrup and Torrilhon [9] that small oscillations at a point results in large oscillations at other points. Bobylev [14] and Struchtrup and Torrilhon [9] showed that the instability of these equations is such that small wavelength fluctuations will blow up in time. Kogan [15] argued that the Burnett equations do not provide solution to the Boltzmann equation in the Knudsen layer. The above mentioned weaknesses have resulted in researchers proposing alternative forms of the Burnett equations (see [7]). While recognizing their weaknesses, Garcia-Colin et al. [8] have recently discussed in detail the usefulness of the Burnett equations. The application of these equations to the Couette flow problem is further explored in this paper. Recently, the authors have been able to analytically solve the Burnett equations for plane Poiseuille flow [16]. Detailed validation of the proposed solution against experimental and DSMC data in the literature showed that the obtained solution is accurate up to $Kn \sim 2.2$, which is higher than all known analytical solutions. Note that this Knudsen number is well within the transition regime.

Lockerby et al. [5], Struchtrup [17], Struchtrup and Torrilhon [11] and Taheri et al. [18] solved the higher-order continuum equations. Lockerby et al. [5] considered the Kramer's problem (gas flow generated by a uniformly applied shear stress and bounded by one parallel surface) and employed additional boundary conditions as suggested by DSMC data to work out the solution for the problem. It was found that different equations predict different thickness of the Knudsen layer – anywhere from 0 to 4.9 times the mean free path for their conditions. Struchtrup [9], Struchtrup and Torrilhon [11] and Taheri et al. [18] argued that the Burnett and super-Burnett equations fail to capture the Knudsen layer correctly; they further showed that the regularized 13-moments equations *qualitatively* capture the Knudsen layer. The present paper extends these earlier works; further comments and comparison with these papers is made in the later sections. The objectives of the present work are: (i) to extend the solution to the transition regime for the plane Couette flow problem, (ii) to provide *quantitative* comparison between the theoretical and DSMC solutions, (iii) to compare solutions from the various higher-order equations, and (iv) to provide an expression for slip velocity. We do not employ any condition at the wall rather evaluate the amount of slip from the model results themselves.

2. Governing equations

In the current paper, we consider the flow between two parallel infinitely long and wide plates, separated by distance H as shown in Fig. 1. The plates move in opposite directions with equal magnitude of velocity ($= U_0/2$). The flow is assumed to be steady, isothermal and incompressible.

2.1. Super-Burnett equations

As shown by Lockerby et al. [5] and Strutchtrup [17], the Burnett equations predict a linear velocity profile; whereas DSMC data suggests that the velocity profile is non-linear at larger values of

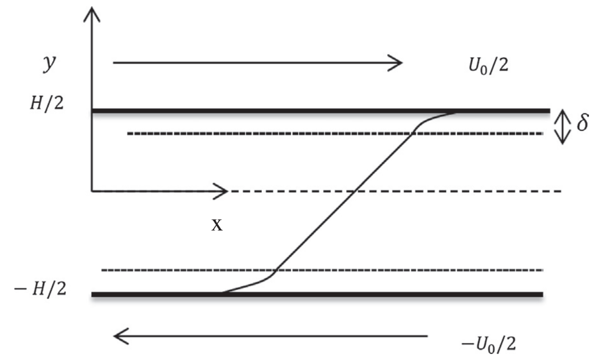

Fig. 1. Schematic of the flow situation considered.

Knudsen number. This shows that the solution cannot be obtained from the Burnett equations; still higher-order equation – the super-Burnett equations, therefore have to be invoked to obtain the solution for this problem.

Starting from the equations in Agarwal et al. [7] and invoking the conditions that $d/dx = 0$ for all quantities (including that for pressure), Eq. (1) given below is obtained. In their approach, the non-linear collision integral in the Boltzmann equation is represented by the BGK model. These equations satisfy the Boltzmann's H-theorem for a wide range of Knudsen numbers [7].

$$-\mu\frac{\partial U}{\partial Y} + \mu^3\frac{RT}{p^2}\frac{\partial^3 U}{\partial Y^3} - \mu^3\frac{1}{p^2}\theta_7\left(\frac{\partial U}{\partial Y}\right)^3 = \tau_{21}(const) \tag{1}$$

where U is gas velocity, ρ is pressure, R is gas constant, T is temperature, μ is viscosity, τ_{21} is shear stress at the walls, and $\theta_7 = -0.4$ is a constant for gas being assumed as hard sphere [7]. The various terms in the equation can be normalized with the characteristic scales:

$$u = \frac{U}{U_0} \quad y = \frac{Y}{H}$$
$$Kn = \frac{\mu}{\rho H}\sqrt{\frac{1}{RT}} \quad \sigma_{21} = \frac{\tau_{21}}{\mu\frac{U_0}{H}} \tag{2}$$

where U_0 is the velocity difference between the two walls. The equation upon non-dimensionalizing reduces to:

$$-\frac{\partial u}{\partial y} + Kn^2\frac{\partial^3 u}{\partial y^3} - Kn^2\gamma Ma^2\theta_7\left(\frac{\partial u}{\partial y}\right)^3 = \sigma_{21} \tag{3}$$

where Ma is Mach number. Note that all equations presented henceforth in the paper are in non-dimensional form unless stated otherwise. Notice that the non-linear term is of $O(Ma^2)$. The non-linear term can therefore be dropped under the condition that we are considering low-speed flow ($Ma < 1/\sqrt{\gamma}$). The simplified equation reads:

$$\sigma_{21} = -\frac{\partial u}{\partial y} + Kn^2\frac{\partial^3 u}{\partial y^3} \tag{4}$$

Shavaliyev [19] first calculated (and Struchtrup [17] later confirmed it) the Super Burnett stress and heat flux terms for one-dimensional and weakly perturbed flows, for gas being assumed as Maxwellian molecules. The Super Burnett terms can be simplified to the following form for the planar Couette flow problem:

$$\sigma_{21} = -\frac{\partial u}{\partial y} - \frac{2}{3}Kn^2\frac{\partial^3 u}{\partial y^3} \tag{5}$$

Both Eqs. (4) and (5) have been considered in this work; these are referred to as super-Burnett [7] and super-Burnett [19] equations, respectively. Zhong et al. [20] suggested adding a higher-order term to the Burnett equations to stabilize the solution. The higher-order