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Improvement of weighted essentially non-oscillatory schemes near discontinuities

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ABSTRACT

In this article, we analyze the fifth-order weighted essentially non-oscillatory (WENO-5) scheme and show that, at a transition point from smooth region to a discontinuity point or vice versa, the accuracy order of WENO-5 is decreased to third order. A new method is proposed to overcome this drawback by introducing fourth-order fluxes combined with high order smoothness indicator. Numerical examples show that the new method is more accurate near discontinuities with accuracy improved to fourth order.

1. Introduction

The WENO scheme concept was first proposed by Liu et al. [1] and then improved by Jiang and Shu [2]. WENO schemes are based on ENO (essentially non-oscillatory) schemes [3,4], but use a convex combination of all candidate stencils instead of the smoothest one as in the ENO schemes. The WENO schemes achieve high order accuracy in smooth regions with more compact stencil and have better convergence due to the smoother numerical flux used.

Jiang and Shu [2] analyze and modify the fifth order WENO scheme proposed by Liu et al. [1] and suggest a new way of measuring the smoothness of a numerical solution. Thus a WENO scheme with the optimal $(2r-1)$th order accuracy rather than $(r+1)$th order is obtained. Henrick et al. [5] point out that the original smoothness indicators of Jiang and Shu fail to improving the accuracy order of WENO scheme at critical points, where the first derivatives are zero. A mapping function is proposed by Henrick et al. [5] to obtain the optimal order near critical points. Borges et al. [6] devise a new set of WENO weights that satisfies the necessary and sufficient conditions for fifth-order convergence given by Henrick et al. [5] and enhances the accuracy at critical points. A class of higher than fifth order weighted essentially non-oscillatory schemes are designed by Balsara and Shu in [7]. Wang and Chen [8] propose optimized WENO schemes for linear waves with discontinuity. Martin et al. [9] suggest a symmetric WENO method by means of a new candidate stencil, which is $2r$th-order accurate and symmetric, and less dissipative than Jiang and Shu's scheme.

The above mentioned WENO schemes are constructed to have $(2r-1)$th or $2r$th [9] order of accuracy in the smooth regions directly from rth order ENO schemes. For a solution containing discontinuities, these methods can not obtain the optimal accuracy near the discontinuity points. Shen et al. [10] indicate that the smoothness indicator IS_k of Jiang and Shu's WENO scheme does not satisfy the condition $\beta_k = D(1+O(\Delta x^2))$ at a critical point $(f_i' = 0)$, and propose a step-by-step reconstruction to avoid the strict condition.

In this article, the analysis of the fifth-order WENO (WENO-5) scheme indicates that, at a transition point from smooth region to a discontinuity point or vice versa, the accuracy order of fifth order WENO scheme is decreased. Two fourth order fluxes are suggested and combined with the higher order smoothness indicators to overcome this drawback. Numerical examples show that this new method is more accurate and achieves higher resolution near discontinuities.

2. Weighted essentially non-oscillatory schemes

For the hyperbolic conservation law in the form

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0 \tag{1}$$

the semi-discretization form can be written as

$$\frac{du_i(t)}{dt} = -\frac{1}{\Delta x}\left(h_{i+\frac{1}{2}} - h_{i-\frac{1}{2}}\right) \tag{2}$$

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The flux $h_{i+\frac{1}{2}}$ of the classical fifth-order WENO scheme [2,6] is built through the convex combination of interpolated values $\hat{f}^k\left(x_{i+\frac{1}{2}}\right)(k=0,1,2)$, in which $\hat{f}^k(x)$ is the third degree interpolation polynomial on stencil $S_k^3 = (x_{i+k-2}, x_{i+k-1}, x_{i+k})$,

$$h_{i+\frac{1}{2}} = \sum_{k=0}^{2} \omega_k \hat{f}^k\left(x_{i+\frac{1}{2}}\right) \tag{3}$$

where

$$\hat{f}^k\left(x_{i+\frac{1}{2}}\right) = \hat{f}^k_{i+\frac{1}{2}} = \sum_{j=0}^{2} c_{kj} f_{i-k+j}, \quad i = 0, \cdots, N \tag{4}$$

The weights ω_k are defined as

$$\omega_k = \frac{\alpha_k}{\sum_{l=0}^{2}\alpha_l}, \quad \alpha_k = \frac{d_k}{(\beta_k + \varepsilon)^p} \tag{5}$$

The smoothness indicators β_k are given by [2]

$$\beta_k = \sum_{l=1}^{2} \Delta x^{2l-1} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(\frac{d^l}{dx^l}\hat{f}^k(x)\right)^2 dx \tag{6}$$

For $r = 3$, Eq. (6) gives

$$\begin{cases} \beta_0 = \frac{13}{12}(f_{i-2} - 2f_{i-1} + f_i)^2 + \frac{1}{4}(f_{i-2} - 4f_{i-1} + 3f_i)^2 \\ \beta_1 = \frac{13}{12}(f_{i-1} - 2f_i + f_{i+1})^2 + \frac{1}{4}(f_{i-1} - f_{i+1})^2 \\ \beta_2 = \frac{13}{12}(f_i - 2f_{i+1} + f_{i+2})^2 + \frac{1}{4}(3f_i - 4f_{i+1} + f_{i+2})^2 \end{cases} \tag{7}$$

Henrick et al. [5] show that if β_k satisfy $\beta_k = D(1 + O(\Delta x^s))$, the weights ω_k then satisfy $\omega_k = d_k + O(\Delta x^s)$, where D is some non-zero quantity independent of k. The necessary and sufficient conditions for fifth-order convergence in (2) are given as [5]:

$$\sum_{k=0}^{2} A_k(\omega^+ - \omega^-) = O(\Delta x^3) \tag{8}$$

$$\omega_k^{\pm} - d_k = O(\Delta x^2) \tag{9}$$

A sufficient condition for fifth-order of convergence is given by Borges et al. in [6]:

$$\omega_k^{\pm} - d_k = O(\Delta x^3) \tag{10}$$

If $f_i' = 0$, Eq. (7) gives $\beta_k = D(1 + O(\Delta x))$ and $\omega_k = d_k + O(\Delta x)$, which degrades the convergence accuracy of the scheme. Shen et al. [10] suggest a step-by-step reconstruction method, in which two fourth order weighted fluxes obtained from 3rd ENO fluxes are used to construct fifth order WENO scheme. Henrick et al. [5] propose a mapping function to increase the approximation of ω_k to the ideal weights d_k.

Borges et al. [6] introduce the absolute difference between β_0 and β_2 to devise a new set of WENO weights that satisfy the necessary and sufficient conditions for fifth-order convergence. The smoothness indicators β_k^z defined by Borges et al. [6] are

$$\beta_k^z = \frac{\beta_k + \varepsilon}{\beta_k + \tau_5 + \varepsilon}, \quad k = 0, 1, 2 \tag{11}$$

and the WENO weights ω_k^z of Borges et al. [6] are

$$\omega_k^z = \frac{\alpha_k^z}{\sum_{l=0}^{2}\alpha_l^z}, \quad \alpha_k^z = \frac{d_k}{\beta_k^z} = d_k\left(1 + \left(\frac{\tau_5}{\beta_k + \varepsilon}\right)^q\right), \quad k = 0, 1, 2 \tag{12}$$

where

$$\tau_5 = |\beta_0 - \beta_2| \tag{13}$$

The coefficients c_{kj} and d_k are listed in Table 1. The parameter ε is used to avoid the division by zero ($\varepsilon = 10^{-6}$ is used in [2] and $\varepsilon = 10^{-40}$ is used in [6]), p and q are chosen to increase the difference of scales of distinct weights at non-smooth parts of the solution. As pointed out by Borges et al. [6], for a smooth function, increasing the value of q in Eq. (12) decreases the correction of the WENO-Z weights to the ideal weights d_k, making the scheme closer to the optimal central scheme. On the other hand, increasing q also decreases the relative importance of the discontinuous substencil and makes the scheme more dissipative.

If $f_i' \neq 0$, Eq. (12) with $q = 1$ gives $\omega_k - d_k = O(\Delta x^3)$; if $f_i' = 0$, (12) with $q = 2$ gives $\omega_k - d_k = O(\Delta x^2)$. The numerical example of Borges et al. [6] shows that, at the first-order critical point ($f_i' = 0$), with $\varepsilon = 10^{-40}$, WENO-JS scheme has third-order accuracy, and WENO-Z scheme with $q = 1$ and $q = 2$ has fourth- and fifth-order accuracy, respectively.

Fifth-order WENO schemes can capture shock waves and have fifth-order accuracy in smooth regions. However, because a WENO scheme is constructed directly from rth-order interpolation to achieve $(2r-1)$th-order, the accuracy is reduced at the transition point from smooth region to discontinuous point and vice versa. In order to illustrate this problem, Fig. 1 is used as an example.

At point $(i-1)$, the stencil $S^5_{(i-1)-1/2}$ is

$$S_{(i-1)^5-1/2} = \{x_{i-4}, x_{i-3}, x_{i-2}, x_{i-1}, x_i\} \tag{14}$$

and it is a smooth stencil, $h_{(i-1)-1/2}$ is obtained by the process of WENO-Z or WENO-JS as a fifth-order flux.

However, for

$$S^5_{(i-1)+1/2} = \{x_{i-3}, x_{i-2}, x_{i-1}, x_i, x_{i+1}\} \tag{15}$$

there is a discontinuity at stencil $S_2^3 = \{x_{i-1}, x_i, x_{i+1}\}$, so

$$\beta_2 \gg \beta_0, \beta_1 \tag{16}$$

no matter whether WENO-Z or WENO-JS is used. To calculate the flux $h_{(i-1)+1/2}$ from either Eq. (5) or (12), it is easy to find

$$\omega_0 \to \frac{1}{7}, \; \omega_1 \to \frac{6}{7}, \; \omega_2 \to 0 \tag{17}$$

The situation at point $(i+3)$ is similar to at the point $i-1$. $S^5_{(i+3)-1/2}$ contains a discontinuity at stencil $S_0^3 = \{x_i, x_{i+1}, x_{i+2}\}$, whereas $S^5_{(i+3)+1/2} = \{x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}, x_{i+5}\}$ is a smooth stencil. For the flux $h_{(i+3)-1/2}$,

$$\omega_0 \to 0, \; \omega_1 \to \frac{2}{3}, \; \omega_2 \to \frac{1}{3} \tag{18}$$

Let us have a look at a numerical example of a discontinuous function [6]

$$u(0,x) = f(x) = \begin{cases} -sin(\pi x) - \frac{1}{2}x^3, & -1 < x \leqslant 0, \\ -sin(\pi x) - \frac{1}{2}x^3 + 1, & 0 < x \leqslant 1, \end{cases} \tag{19}$$

consisting of a piecewise *Sine* function with a jump discontinuity at $x_i = 0$. The weights calculated by WENO-Z scheme (Eq. (12)) is shown in Fig. 2, it demonstrates the accuracy degrading problem. For the flux $h_{(i-1)+1/2}, \omega_0 \approx \frac{1}{7}$ (point A), $\omega_1 \approx \frac{6}{7}$ (point B). For $h_{(i+3)-1/2}, \omega_1 \approx \frac{2}{3}$ (point D), $\omega_1 \approx \frac{1}{3}$ (point C).

Under the condition of $\Delta x \to 0$, there are

$$h_{(i-1)-\frac{1}{2}} = \frac{1}{30}f_{i-4} - \frac{13}{60}f_{i-3} + \frac{47}{60}f_{i-2} + \frac{9}{20}f_{i-1} - \frac{1}{20}f_i \tag{20}$$

Table 1
Coefficients c_{kj} and d_k.

c_{kj}				d_k
k	j = 0	j = 1	j = 2	
0	1/3	−7/6	11/6	1/10
1	−1/6	5/6	1/3	6/10
2	1/3	5/6	−1/6	3/10