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An immersed interface solver for the 2-D unbounded Poisson equation and its application to potential flow

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ABSTRACT

This paper presents a novel algorithm to solve the 2-D potential flow past complex geometries with circulation in unbounded domain and in the presence of a given vorticity field. It is based on a Poisson solver that combines two components: the immersed interface method to enforce the boundary condition on each inner boundary and the James–Lackner algorithm to compute the outer boundary condition consistent with the unbounded domain solution. The algorithm is here based on second order finite differences and it requires solely 1-D stencil corrections; this makes the immersed interface part of the present method easily extendable to higher dimensional problems. The treatment of the outer boundaries requires an iterative boundary potential method. The algorithm is validated, by means of grid convergence studies, on the flow past multiple bodies. The results confirm the second order accuracy everywhere. The algorithm is also self consistent as "all is done on the grid" (thus without using a vortex panel boundary element method in addition to the grid). For cusped airfoils, a consistent way to enforce the *Kutta–Joukowsky condition* is also presented. The present algorithm constitutes a crucial building block towards an immersed interface-enabled vortex particle-mesh method for the computation of unsteady viscous flows, with boundary layers, detached shear layers and wakes. A possible extension to 3-D problems is also briefly discussed.

1. Introduction

The study and the development of solution techniques for Poisson equations is a recurring research topic as they appear in many areas of mathematical and computational physics, e.g. electromagnetism, continuum mechanics and theoretical physics. This wide range of applications has given rise to many different solution techniques. We here consider two aspects of the problem that still remain a challenge today: taking into account irregular interior boundary geometries and providing outer boundary conditions that are compatible with the solution of the equation in an unbounded domain.

These two key components can also be found in a more specific context, in the framework of Computational Fluid Dynamics for the simulation of the flow past bluff bodies in an unbounded domain (external flow aerodynamics). Moreover, in incompressible fluid dynamics, one is always constrained to solve at least one Poisson equation per time step and obtaining its solution represents the most expensive computational step. The choice of the present application, namely potential flow in the presence of a given vorticity field, follows this observation and is motivated by the fact that it represents one of the computational steps required for the simulation of unsteady bluff body flows using a viscous vortex particle method combined with a vortex panel method [1] (boundary element method), as explained for example in [2,3].

Whether for Poisson equations, for the Navier–Stokes equations in fluid dynamics, or for other types of PDE's, great efforts have been made in order to incorporate irregular boundary geometries inside the so-called structured grid methods (finite difference methods, spectral methods, etc.).

One of the first attempts to achieve this goal in the context of fluid dynamics was undertaken by Peskin [4]. It is considered to be the pioneering work for a class of methods known as the immersed boundary methods [5,6]. This class of methods provides a discrete representation of the singular source term acting at the irregular boundary which is immersed inside the computational domain. Hence, considering the flow past moving bodies which are either rigid or even deforming is greatly simplified as the grid must not be adapted to fit the boundaries.

Based on a similar approach and following the same goal, Brinkman-type penalization methods have also emerged [7]. The latter approach can also be applied in combination with different kinds of discretization methods, i.e. spectral methods [8], finite

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differences [7] or vortex particle methods [9–11]. However, the regularization of the singular source term over a few grid cells entails a smearing of the solution near the interface, as has been shown for example in [12], and can lead to a loss of accuracy [13]. For high Reynolds number flows, this can be problematic because the boundary layers may not be captured properly. Therefore, methods capturing sharp interfaces have been developed, such as the ghost-cell approach [14], the cut-cell method [15] or the hybrid Cartesian/immersed boundary method (HCIB [16,17]).

In the same spirit, immersed interface methods have appeared in the literature [18] as a consistent way to take into account possible jumps of the unknown at the interface, e.g. by modifying the finite difference stencil in the vicinity of the interface. The original stencil correction technique [18] uses multiple-dimensional Taylor series. This can however lead to stability issues in the resulting linear system resolution and requires among others a careful choice of the stencil nodes [21]. Other methods use instead one-dimensional Taylor series (*dimension splitting* approach, see [19,12,20]). The method developed here is based on the latter approach from [12], as the stencil correction procedure is more easily applicable to higher dimensional problems: for each grid direction, the stencil corrections are derived at the intersections of the interface with the different grid axes along the corresponding direction, according to the prescribed boundary condition. The present approach thus provides a special treatment for grid nodes close to the interface. This feature is shared by all sharp-interface capturing methods. The cut-cell method modifies the grid cell geometry near the interface, the ghost-cell approach extends the solution across the boundary and the HCIB method interpolates the solution on the interior grid nodes closest to the interface using the solution and the boundary condition.

The other key component considered here is the unbounded outer boundary condition. The most natural way to take this into account for a Poisson equation is to perform the convolution of the source term with the free space Green's function, either through direct summation and ideally by fast summation (fast multipole method in two [22] and three dimensions [23]). Another class of methods is based on fast Fourier transforms [24,25] but, as the immersed interface approach requires local modifications of the spatial differential operator, it is hardly applicable here.

Therefore, we follow a different approach based on the James–Lackner algorithm [26,27], which has been further improved in [28] and which additionally remains compatible with mesh refinement techniques. The solution procedure is based on two problems, the first one being obtained by imposing homogeneous Dirichlet conditions on the outer boundary and the second problem computes correction charges at the outer boundary which result in an inhomogeneous Dirichlet condition being consistent with the unbounded character of the solution. Miller [29] extended the method to include some irregular interior boundaries held at a fixed potential. The presence of interior boundaries with unknown surface charges results in a method which is intrinsically iterative.

The present approach combines the work of Linnick and Fasel [12] and Miller [29] and generalizes the algorithm to allow the computation of potential flow past multiple bodies accounting for a given compact vorticity field. In this case, the stream function is the superposition of a function linear in space (free stream flow field) and an unbounded solution of the Poisson equation. The stream function solution is constant in the interior boundaries but the value of this constant is not known a priori. This value is determined by a supplementary constraint about the circulation of the flow around each solid body.

In this paper, we thus propose a second-order finite difference method to compute the solution of a two-dimensional Poisson equation in an unbounded domain with interior boundaries of complex geometry.

The underlying objective of the present work is to integrate the resulting method [30,31]. Vortex particle methods perform very well for unbounded vortical flows but accounting for solid bodies remains difficult. Penalization methods have been used (as mentioned above [9–11]). A different technique consists in combining the Poisson solver with a boundary element method to account for the presence of the walls, either by combining it with a vortex panel method [2,3,32,33], either by computing equivalent sources of velocity potential [34,35]. This procedure allows to recover from a given vorticity field a velocity field that also respects the no through-flow condition at the surface of the body. In the specific context of vortex particle-mesh (VPM) methods, relying simultaneously on particles and on a grid [36,37,33], the present approach is a more consistent alternative to the combination of the finite difference Poisson solver with the boundary element method, as it preserves the order of convergence up to the wall.

In Section 2, the governing equations for the elliptic problem are given. Section 3 is devoted to the methodology description: in Section 3.1, the immersed interface approach is detailed in order to take into account the interior boundaries with prescribed outer boundary conditions; in Section 3.2, the iterative boundary potential method is detailed so as to obtain the correct outer boundary conditions; in Section 3.3, the global algorithm is given, and Section 3.4 briefly presents a possible extension of the approach to three-dimensional problems. Section 4 is devoted to the validation of the methodology for several potential flow problems.

Results are first compared with the analytical solution for the flow past a cylinder. The convergence behavior of the approach is assessed and the error value is compared with that obtained using a vortex panel method. A convergence study is also performed for the prediction of the added mass coefficient of an elliptical cylinder. Next, the order of convergence is assessed for the flow past an airfoil with a cusped trailing edge. This case requires the development of a supplementary equation to enforce the *Kutta–Joukowsky condition*. Finally, the ability of the method to take into account multiple bodies as well as more general geometries is also illustrated and validated.

2. Problem statement

In many applications of computational fluid dynamics, the solution of a Poisson equation is required. In particular, the operation of computing a velocity field $\mathbf{u}$ associated to a given vorticity field $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is required when working with the velocity–vorticity formulation of the Navier–Stokes equations for incompressible flows ($\nabla \cdot \mathbf{u} = 0$)

$$\frac{D\boldsymbol{\omega}}{Dt} \triangleq \frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = (\nabla \mathbf{u}) \cdot \boldsymbol{\omega} + \nu \nabla^2 \boldsymbol{\omega} \quad \nabla^2 \boldsymbol{\Psi} = -\boldsymbol{\omega},$$

with ν the kinematic viscosity of the fluid. Indeed, the velocity field $\mathbf{u}$ can be linked to the vorticity $\boldsymbol{\omega}$ through the above Poisson equation for the streamfunction $\boldsymbol{\Psi}$, as $\mathbf{u} = \nabla \times \boldsymbol{\Psi}$ and $\nabla \cdot \boldsymbol{\Psi} = 0$ (Lorenz' gauge).

The flow past a nonmoving body with boundary $\partial\Omega_b$ is sketched in Fig. 1, in the 2-D case where $\boldsymbol{\Psi} = \Psi\hat{\mathbf{z}}$ and $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$. The flow domain is then $\Omega_f \triangleq \mathbb{R}^2/\Omega_b$ and the boundary conditions are $\lim_{|\mathbf{x}|\to\infty}\mathbf{u} = \mathbf{U}_\infty$ (with $\mathbf{U}_\infty$ a constant free stream flow) and $\mathbf{u} = 0$ on $\partial\Omega_b$ (no slip condition). The translation of the no slip condition into vorticity formulation is not straightforward. Usually, the Poisson equation is solved with a *no through-flow* condition on $\partial\Omega_b$. This is actually the core of the present work and we refer to the problem as finding the *potential flow* that cancels the through flow induced by the vorticity field. The potential velocity field however still presents a residual tangential slip velocity at the wall and the way to enforce a no slip condition based on this is further detailed in [2,3].